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FOURIER-STIELTJES TRANSFORMS AND SINGULAR INFINITE CONVOLUTIONS.*

By NORBERT WIENER and AUREL WINTNER.

1. Let $F(x)$, $-\pi \leqq x \leqq \pi$, be a continuous monotone function and

$$(1) \qquad c_n = c_n(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}dF(x) = \bar{c}_{-n} \qquad (n = 0, 1, 2, \cdots)$$

the sequence of its trigonometrical moments or Fourier-Stieltjes coefficients. If $F(x)$ is absolutely continuous, i. e., if

$$(2) \qquad F(x) = \int^{x} f(y)dy \qquad (-\pi \leqq x \leqq \pi)$$

holds for an L-integrable function f, then

$$(3) \qquad c_n = o(1)$$

by the Riemann-Lebesgue lemma. The converse is not true, since an example of Menchoff [8] shows that (3) may hold for a monotone function $F(x)$ which is singular, i. e., continuous and such that its derivative vanishes almost everywhere but $F(x) \neq \text{const.}$

Let an absolute constant κ^* be defined by means of a Dedekind cut as follows: For a given number κ, there do or do not exist singular monotone functions $F(x)$ which satisfy

$$(4) \qquad c_n = o(n^{-\kappa})$$

according as $\kappa < \kappa^*$ or $\kappa > \kappa^*$. Menchoff's result, when expressed in terms of κ^*, does not state more than that κ^* cannot be negative, a fact which is implied by $c_n = O(1)$ and is, therefore, clear from (1). Also a result of Kershner [6], which has improved on Menchoff's example (3), does not give more than $\kappa^* \geqq 0$. However, Littlewood [7] has recently proved by an ingenious, though rather involved, construction that $\kappa^* \neq 0$. Littlewood's proof of $\kappa^* > 0$ is a pure existence proof, since it seems to be hardly possible to prove by his method that $\kappa^* > 10^{-10}$, say. At any rate, Littlewood's result $\kappa^* > 0$ implies the existence of a critical κ-range within which (4) is incapable

* Received May 9, 1938.

of deciding whether or not $F(x)$ is absolutely continuous. In fact, a well-known remark of Lebesgue shows that the Riemann-Lebesgue lemma, i. e. (3), cannot be improved for the class of all absolutely continuous F (not even if f in (2) is restricted to be continuous and periodic).

On the other hand, the critical κ-range is bounded, i. e., $\kappa^* < +\infty$. Actually, $\kappa^* \leqq \frac{1}{2}$. In fact, if F is such that (4) holds for a $\kappa > \frac{1}{2}$, then $\Sigma \mid c_n \mid^2 < +\infty$, and so the Fischer-Riesz theorem assures that (2) holds for a function f which is of class L, since it is of class L^2.

2. Accordingly, $0 < \kappa^* \leqq \frac{1}{2}$. No better result (e. g., $10^{-10} \leqq \kappa^* \leqq \frac{1}{2} - 10^{-10}$) is known to us from the existing literature. But we shall prove that $\kappa^* = \frac{1}{2}$. In other words, we shall show that there exists for every $\epsilon > 0$ a singular monotone function $F = F_\epsilon$ such that

$$c_n = O(n^{-\frac{1}{2}+\epsilon}). \tag{5}$$

It seems to be a reasonable guess that if $\gamma_1, \gamma_2, \cdots$ are given positive numbers for which $\Sigma \gamma_n{}^2 = +\infty$, then there exists a singular monotone function F such that $c_n = O(\gamma_n)$ or $c_n = o(\gamma_n)$.

It will remain undecided whether or not the width, $\kappa^* = \frac{1}{2}$, of the critical κ-range is directly connected with the critical widths $\frac{1}{2}$ in the theory of ordinary Dirichlet series (although the method will be such as to prove (5) also when n in (1) is allowed to vary continuously). Actually, it will be seen that the absolute constant κ^* is the "same" $\frac{1}{2}$ which occurs in the standard asymptotic formulae of the type

$$J_0(u) = O(\mid u \mid^{-\frac{1}{2}}), \qquad J_0(u) = \Omega(\mid u \mid^{-\frac{1}{2}}).$$

In fact, the exponent $-\frac{1}{2}$ of (5) will be introduced by using van der Corput's lemma,

$$\mid \int_a^b \cos g(x)\, dx \mid \leqq \text{Const.} \min_{a \leqq x \leqq b} \mid g''(x) \mid^{-\frac{1}{2}}, \tag{6}$$

where $g(x)$, $a \leqq x \leqq b$, is any real function with a non-vanishing continuous second derivative $g''(x)$, and Const. is independent of a, b and g (cf. Kershner [5]).

The proof of $\kappa^* = \frac{1}{2}$ will consist of four steps:

(i) It turns out (§ **5**) that, instead of proving the existence of a singular monotone $F(x) = F_\epsilon(x)$ for which the $c_n = c_n(F)$ are small in the sense of (5), it is sufficient to prove the existence of a singular monotone $G(x) = G_\epsilon(x)$ for which the $c_n = c_n(G)$ are small on the *average*, in the sense that

$$\sum_{n=-N}^{N} | c_n | = O(N^{\epsilon}) \qquad (N \to +\infty). \tag{7}$$

(ii) It will be shown (§ **5**) that an $F(x)$, $-\pi \leqq x \leqq \pi$, which satisfies (5) can always be derived from a $G(x)$, $-\pi \leqq x \leqq \pi$, which satisfies (7), the passage from G to F consisting of an elementary one-to-one continuous transformation of the interval $-\pi \leqq x \leqq \pi$ into itself. It is only through this transformation that (5) becomes satisfied, since before this passage from G to F not even (3) is satisfied (cf. (29), § **7**). The effect of this transformation on the Fourier-Stieltjes coefficients may be estimated by means of the second mean value theorem, which introduces into (5) the exponent $-\frac{1}{2}$ of (6).

(iii) The theory of symmetric Bernoulli convolutions (cf. Jessen and Wintner [3], Kershner and Wintner [4], and Wintner [10], where further references are given) implies that the construction of a G which satisfies (7) can be based (§ **7**) on suitable estimations of the function

$$L(u) = \prod_{k=1}^{\infty} \cos (3^{-k}u) \qquad (-\infty < u < +\infty) \tag{8}$$

for large $| u |$. Since $\cos v = 1 + O(v^2)$ as $v \to 0$, the product (8) defines an entire function of the exponential type. Actually, (8) is the Fourier-Stieltjes transform of the symmetric Bernoulli convolution which defines the classical Cantor function (cf. Carleman [1], pp. 223-226). Thus, while singular Bernoulli convolutions will occur, they will be used for quite another purpose than in the examples of Menchoff [8] and Kershner [6], examples which are in themselves singular Bernoulli convolutions and satisfy (3).

(iv) The remaining step, namely that estimate of (8) which was mentioned under (iii), will be made (§ **6**) by an adaptation of an enumeration process which is familiar in a similar, though easier, instance; cf. Wiener [9].

All of this will apply also for Fourier-Stieltjes transforms (§ **8**), instead of trigonometrical moments; so that in (1) the index n can be replaced by a continuous variable u.

3. Let $\xi = \xi(x)$, $-\pi \leqq x \leqq \pi$, be a strictly increasing continuous function such that $\xi(-\pi) = -\pi$, $\xi(\pi) = \pi$. Then, if $G(x)$, $-\pi \leqq x \leqq \pi$, is any continuous monotone function, so is the function $F(x)$, $-\pi \leqq x \leqq \pi$, which is defined by $G(x) = F(\xi(x))$, i. e., by $F(x) = G(\xi^{-1}(x))$. Clearly,

$$\int_{-\pi}^{\pi} e^{-imx} dF(x) = \int_{-\pi}^{\pi} e^{-imx} dG(\xi^{-1}(x)) = \int_{-\pi}^{\pi} e^{-im\xi(x)} dG(x).$$

Hence, on placing

$$\lambda_{nm} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\xi(x)} e^{inx}\,dx, \qquad (m, n = 0, \pm 1, \cdots), \tag{9}$$

and developing the function $e^{-im\xi(x)}$, which is of bounded variation for every fixed m, into its Fourier series

$$e^{-im\xi(x)} = \sum_{n=-\infty}^{+\infty} \lambda_{nm} e^{-inx}, \qquad (m = 0, \pm 1, \cdots),$$

finally applying, for instance, a standard theorem of W. H. Young, one clearly obtains

$$c_m(F) = \sum_{n=-\infty}^{+\infty} \lambda_{nm} c_n(G), \qquad (m = 0, \pm 1, \cdots), \tag{10}$$

if use is made of the notation (1).

Suppose, in particular, that the topological transformation $\xi = \xi(x)$ of the interval $[-\pi, \pi]$ into itself is such that the derivative of the function $\xi(x)$, or the distortion of the mapping $\xi = \xi(x)$, lies between two positive bounds. In this case it is clear that a monotone continuous function $G(x)$ is singular if and only if the same holds for the function $F(x) = G(\xi^{-1}(x))$.

4. Let

$$\xi = \xi(x) \equiv \tfrac{1}{4}(3x + \tfrac{1}{\pi} x^2 \operatorname{sgn} x), \quad \text{where} \quad -\pi \leqq x \leqq \pi. \tag{11}$$

This $\xi(x)$ is continuous, strictly monotone and such that $\xi(-\pi) = -\pi$, $\xi(\pi) = \pi$. The coefficients (9) belonging to the mapping (11) are

$$\lambda_{nm} = \frac{1}{\pi}\int_{0}^{\pi} \cos\{\tfrac{1}{4}m(3x + \tfrac{1}{\pi} x^2) - nx\}\,dx. \tag{12}$$

Hence, from (6),

$$|\lambda_{nm}| < \text{Const.}\, |m|^{-\frac{1}{2}} \text{ for all } n, m. \tag{13}$$

It will be shown that

$$|\lambda_{nm}| < \text{Const.}\, n^{-2} \text{ whenever } |n| > 2\,|m|. \tag{14}$$

First, it is easily verified by differentiation with respect to x that if $|n| > 2\,|m|$, the function $\{\ \}$ whose cosine is the integrand of (12) is strictly monotone for $0 \leqq x \leqq \pi$. Hence, on placing $\{\ \} = t$ and

$$f_{nm}(t) = |(\tfrac{3}{4}m - n)^2 + \tfrac{1}{\pi} mt|^{-\frac{1}{2}}, \tag{15}$$

the integral (12) can be written as

$$\lambda_{nm} = \pm \frac{1}{\pi} \int_0^{(m-n)\pi} f_{nm}(t) \cos t \, dt.$$

Consequently, on applying partial integration and noting that $\frac{1}{\pi} < 1$,

$$|\lambda_{nm}| \leqq \left| \int_0^{(m-n)\pi} f'_{nm}(t) \sin t \, dt \right|, \text{ where } ' = d/dt.$$

Now, it is easily verified from the assumption $|n| > 2|m|$ that the derivative, $f'_{nm}(t)$, of the function (15) is monotone between $t = 0$ and $t = (m - n)\pi$. Consequently, application of the second mean value theorem to the last integral gives

$$|\lambda_{nm}| \leqq 2|f'_{nm}(0)| + 2|f'_{nm}((m - n)\pi)|.$$

Hence, on calculating $f'_{nm}(0)$ and $f'_{nm}((m - n)\pi)$ from (15) and then using the assumption $|n| > 2|m|$, the appraisal (14) follows.

Incidentally, (13) also is a consequence of the second mean value theorem, since so is (6).

5. Writing (10) in the form

$$c_m(F) = \{ \sum_{n=-\infty}^{-2|m|-1} + \sum_{n=-2|m|}^{2|m|} + \sum_{n=2|m|+1}^{+\infty} \} \lambda_{nm} c_n(G),$$

(13) and (14) imply that, as $m \to \pm \infty$,

$$c_m(F) = O(\sum_{n=-\infty}^{-2|m|} n^{-2}) + O(\sum_{n=-2|m|}^{2|m|} |c_n(G)| \, m^{-\frac{1}{2}}) + O(\sum_{n=2|m|}^{+\infty} n^{-2}),$$

since $|c_n(G)| \leqq c_0(G) = \text{const.}$, by (1). Accordingly,

$$c_m(F) = O(|m|^{-1}) + |m|^{-\frac{1}{2}} O(\sum_{n=-2|m|}^{2|m|} |c_n(G)|) \text{ as } m \to \pm \infty.$$

Hence, if $G(x)$ satisfies (7) for a fixed ϵ, then $F(x)$ satisfies (5) for the same ϵ. Furthermore, application of the last remark of § **3** to the mapping (11) shows that both monotone functions $F(x)$, $G(x)$ are singular if either of them is. This proves (i)-(ii), § **2**.

6. Next, (iv), § **2** will be established, by showing that there exists for every $\epsilon > 0$ a sufficiently large positive integer $p = p_\epsilon$ such that for the function (8) one has

$$\sum_{n=-N}^{N} | L(2\pi n)|^p = O(N^\epsilon) \text{ as } N \to +\infty, \tag{16}$$

and even

$$\sum_{n=-N}^{N} | L(\alpha n)|^p = O(N^\epsilon) \text{ uniformly for } \pi \leqq \alpha \leqq 2\pi, \text{ as } N \to +\infty. \tag{17}$$

First, if n is a positive integer, let $\nu = \nu_n$ denote the number of digits in the triadic representation of n; so that

$$n = \delta_{\nu+1}\delta_\nu \cdots \delta_k \cdots \delta_2\delta_1, \qquad (\nu = \nu_n), \tag{18}$$

where every δ_k is either 0 or 1 or 2, while $\delta_\nu \neq 0$ and $\delta_{\nu+1} = 0$. Clearly, the fractional part, $\beta - [\beta]$, of $\beta = 3^{-k}n$ is

$$0.\,\delta_k\delta_{k-1} \cdots \delta_2\delta_1 \qquad (k = 1, \cdots, \nu_n).$$

Hence, on considering the six possible pairs

$$\{\delta_k; \delta_{k-1}\} = \{0;1\},\ \{1;0\},\ \{1;2\},\ \{2;1\},\ \{2;0\},\ \{0;2\},$$

it is easily verified that the fractional part of $3^{-k}n$ either lies between $1/9$ and $4/9$ or between $5/9$ and $8/9$. Consequently, in both cases,

$$|\cos(3^{-k}2\pi n)| \leqq \cos(\pi/9) < \exp(-1/100).$$

Since this holds for $k = 1, \cdots, \nu_n$, and since every factor of the product (8) lies between -1 and 1, it follows that

$$| L(2\pi n)| < \exp(-\mu_n/100),$$

where μ_n denotes the number of changes of the digits δ in the finite sequence (18), i.e., the number of those k for which $\delta_{k+1} \neq \delta_k$. Thus, if $p > 0$,

$$| L(2\pi n)|^p < \exp(-p\mu_n/100). \tag{19}$$

Now, if m is a positive integer, the number of those among the first N positive integer n which have exactly m changes of digits in their triadic representation (18), i.e., the number of those n for which $1 \leqq n \leqq N$ and $\mu_n = m$, cannot exceed

$$(6\nu_N)^m/m!,$$

six being the number of distinct pairs of digits 0, 1, 2. It follows, therefore, from (19) that if the positive integer p is fixed and $N \to +\infty$, then

$$\sum_{n=1}^{N} | L(2\pi n)|^p = O\{\sum_{m=0}^{\infty} \exp(-pm/100) \cdot (6\nu_N)^m/m!\}.$$

Since (18) clearly implies that

$$\nu_n = O(\log n) \text{ as } n \to +\infty,$$

it follows that, if p is fixed and $N \to +\infty$,

$$\sum_{n=1}^{N} | L(2\pi n) |^p = \sum_{m=0}^{\infty} \{\exp(-p/100) \cdot O(\log N)\}^m / m! \, .$$

Consequently, there exists for every $\epsilon > 0$ a sufficiently large integer $p = p_\epsilon > 0$ such that, as $N \to +\infty$,

$$\sum_{n=1}^{N} | L(2\pi n) |^p = O\{ \sum_{m=0}^{\infty} (\epsilon \log N)^m / m! \} = O\{\exp(\epsilon \log N)\},$$

and so, by (8),

$$\sum_{n=-N}^{N} | L(2\pi n) |^p = 1 + 2O\{\exp(\epsilon \log N)\} = O(N^\epsilon).$$

This proves (16). And (17) is proved in the same way, if, starting with an arbitrary positive n which need not be an integer, one applies (18) to the integral part $[n]$ of n, instead of to n. Then (19) remains valid if the subscript n of μ_n is replaced by $[n]$; so that the balance of the above proof can be repeated without change.

7. All that remains to be done is what is assigned by (iii), § **2**.

If $H = H(x)$, $-\infty < x < +\infty$, is a distribution function, (i.e., a monotone function such that $H(-\infty) = 0$, $H(+\infty) = 1$), let $\Lambda(u; H)$ denote its Fourier-Stieltjes transform,

$$\Lambda(u; H) = \int_{-\infty}^{+\infty} e^{iux} dH(x); \qquad -\infty < u < +\infty. \tag{20}$$

Then

$$\Lambda(u; H_{\mathrm{I}} * H_{\mathrm{II}}) = \Lambda(u; H_{\mathrm{I}}) \Lambda(u; H_{\mathrm{II}}), \tag{21}$$

if $H_{\mathrm{I}} * H_{\mathrm{II}}$ denotes the convolution

$$\int_{-\infty}^{+\infty} H_{\mathrm{I}}(x - y) \, dH_{\mathrm{II}}(y), \tag{22}$$

which is a distribution function for any pair H_{I}, H_{II} of distribution functions. It is implied by (21) that

$$H_{\mathrm{I}} * H_{\mathrm{II}} = H_{\mathrm{II}} * H_{\mathrm{I}}, \qquad (H_{\mathrm{I}} * H_{\mathrm{II}}) * H_{\mathrm{III}} = H_{\mathrm{I}} * (H_{\mathrm{II}} * H_{\mathrm{III}}),$$

and that, for any positive integer p,

$$(23)\qquad \Lambda(u; H^{(p)}) = (\Lambda(u; H))^p,$$

if, starting with an H, one puts

$$(24)\qquad H^{(1)} = H,\quad H^{(2)} = H^{(1)} * H,\quad H^{(3)} = H^{(2)} * H, \cdots.$$

Let, in particular, $H(x)$ be the distribution function which is 0 if $x \leqq -\frac{1}{2}$ and 1 if $x \geqq \frac{1}{2}$, while if $|x| \leqq \frac{1}{2}$, then $H(x)$ is the classical Cantor function, obtained in the usual manner by successive trisections of the interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$. Then $H(x)$ is, for $-\infty < x < +\infty$, the infinite convolution

$$H(x) = B(3x) * B(3^2 x) * \cdots * B(3^k x) * \cdots,$$

where $B(x)$ is the distribution function which has a jump of $\frac{1}{2}$ at $x = -1$ and $x = 1$; so that $\Lambda(u; B) = \cos u$. Correspondingly,

$$(25)\qquad \Lambda(u; H) = \prod_{k=1}^{\infty} \cos(3^{-k} u).$$

Furthermore, from (24),

$$(26)\qquad H^{(p)}(x) = B(a_{1p}x) * B(a_{2p}x) * \cdots * B(a_{kp}x) * \cdots$$

where

$$(27)\qquad a_{kp} = 3^j, \text{ if } (j-1)p < k \leqq jp, \text{ where } j = 1, 2, \cdots.$$

Also, from (23), (25) and (8),

$$(28)\qquad \Lambda(u; H^{(p)}) = (L(u))^p.$$

Since it is clear from (8) that $L(3^h \cdot 2\pi)$ is independent of h for $h = 1, 2, \cdots$, and that $L(6\pi) > 0$, it follows from (28) that

$$(29)\qquad \Lambda(3^h \cdot 2\pi; H^{(p)}) = (L(6\pi))^p > 0 \text{ for } h = 1, 2, \cdots.$$

Now, the distribution function $H(x)$ is, if $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$, the Cantor function, so that $H(-\frac{1}{2}) = 0$, $H(\frac{1}{2}) = 1$, and so $H(x)$ is continuous for $-\infty < x < +\infty$. Hence it is clear from the definition (20) of a convolution that each of the iterated functions (24) is continuous for $-\infty < x < +\infty$. On the other hand, $H^{(p)}(x)$ is, in view of (26), an infinite convolution of step functions. Consequently (cf. Jensen and Wintner [3], Theorem 35), the distribution function $H^{(p)}(x)$ either is singular or absolutely continuous. Since (29) assures that $\Lambda(u; H^{(p)}) \to 0$, $u \to \pm\infty$, is impossible for a fixed p, it follows that $H^{(p)}(x)$ is singular for every p.

Furthermore, $H(x)$ is constant outside the interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$. Hence

it is clear from (27) and from the definition (22) of a convolution that $H^{(p)}(x)$ is constant outside the interval $-\frac{1}{2}p \leqq x \leqq \frac{1}{2}p$. Accordingly, by (20),

$$\Lambda(u; H^{(p)}) = \int_{-\frac{1}{2}p}^{\frac{1}{2}p} e^{iux} dH^{(p)}(x).$$

It follows, therefore, from (28) and (17) that there exists for every $\epsilon > 0$ a sufficiently large $p = p_\epsilon$ such that, as $N \to +\infty$,

$$\sum_{n=-N}^{N} \left| \int_{-\frac{1}{2}p}^{\frac{1}{2}p} e^{i\alpha n x} dH^{(p)}(x) \right| = O(N^\epsilon)$$

holds uniformly for $\pi \leqq \alpha \leqq 2\pi$.

Thus, on keeping ϵ fixed (hence also $p = p_\epsilon$), and then changing the unit along the x-axis in such a way that the interval $-\frac{1}{2}p \leqq x \leqq \frac{1}{2}p$ becomes the interval $-\pi \leqq x \leqq \pi$, one has

$$(30) \qquad \sum_{n=-N}^{N} \left| \int_{-\pi}^{\pi} e^{inx} dG(x) \right| = O(N^\epsilon),$$

if $G = G_\epsilon$ denotes the monotone function which results from $H^{(p)}$, where $p = p_\epsilon$, upon the change of units. Since $H^{(p)}$ is singular, so is G. Hence, comparison of (30) and (1) assures the existence of a singular monotone G which satisfies (7).

This completes the proof of (5), i. e., of $\kappa^* = \frac{1}{2}$.

8. While the Cantor function $H(x)$ is constant almost everywhere, (27) implies that the singular monotone function (26) is, if $p > 1$, nowhere constant on an interval (cf. Kershner and Wintner [4], p. 547). Consequently, the above result implies one recently proved by Hartman and Kershner [2].

Actually, the above constructions can easily be extended to the case in which the sequence of Fourier-Stieltjes coefficients is replaced by the corresponding Fourier-Stieltjes transform. To this end, all that is needed is to replace in (10) the summation by an integration, and then, with the help of (17), to repeat the above proof with obvious modification. Then it follows that there exists for every $\epsilon > 0$ a singular monotone function $F(x)$, $-\pi \leqq x \leqq \pi$, such that

$$(31) \qquad \int_{-\pi}^{\pi} e^{iux} dF(x) = O(|u|^{-\frac{1}{2}+\epsilon}) \quad \text{as} \quad u \to \pm\infty,$$

where u is a continuous variable. It is also seen that (31) can be chosen as an even function, i. e., such that the curve $y = F(x)$ is symmetric with respect to the line $y = F(0)$.

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ON THE CHARACTERIZATION OF A DISTRIBUTION FUNCTION BY ITS FOURIER TRANSFORM.*

By R. P. BOAS, JR.[1] and F. SMITHIES.

1. By a distribution function we shall mean a non-decreasing function $\sigma(x)$ defined on $-\infty < x < \infty$ such that $\sigma(-\infty) = 0$, $\sigma(\infty) = 1$. The value of $\sigma(x)$ at a point of discontinuity will be assumed to be fixed by the normalization $\sigma(x) = \frac{1}{2}[\sigma(x+0) + \sigma(x-0)]$.

If $\sigma(x)$ is a distribution function, its Fourier transform $\Lambda(u;\sigma)$ is defined by

$$(1.1) \qquad \Lambda(u;\sigma) = \int_{-\infty}^{\infty} e^{iux} d\sigma(x) \qquad (-\infty < u < \infty).$$

If $\Lambda(u;\sigma)$ is given, $\sigma(x)$ is determined by the inversion formula[2]

$$(1.2) \qquad \sigma(x) - \sigma(0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1 - e^{-iux}}{u} \Lambda(u;\sigma) du,$$

where the integral is to be interpreted, if necessary, as a Cauchy principal value.

It is well known that there is a correlation between the smoothness of $\sigma(x)$ and the order of magnitude of $\Lambda(u;\sigma)$ as $|u| \to \infty$. For example, if

$$\Lambda(u;\sigma) = O(|u|^{-k}) \qquad (|u| \to \infty)$$

for every $k > 0$, $\sigma(x)$ is of class C^∞ in $(-\infty, \infty)$; again, if

$$(1.3) \qquad \Lambda(u;\sigma) = O(e^{-c|u|}) \qquad (|u| \to \infty)$$

for some $c > 0$, $\sigma(x)$ is analytic on $(-\infty, \infty)$, and, furthermore, there is a function $\sigma(z)$ of the complex variable $z = x + iy$, such that $\sigma(z)$ is analytic and bounded in every strip $|y| \leqq c' < c$ and coincides with $\sigma(x)$ on the real axis.

It is known that if (1.3) is replaced by the weaker condition

$$\Lambda(u;\sigma) = O\{\exp(-c|u|^{\alpha})\} \qquad (|u| \to \infty)$$

for some $\alpha < 1$, $\sigma(x)$ is not necessarily analytic on the whole real axis; for

* Received April 11, 1938.

[1] National Research Fellow.

[2] See, e. g., S. Bochner, *Vorlesungen über Fouriersche Integrale*, 1932, p. 66.

example, the function $\exp(-c\,|\,u\,|^{\alpha})$ $(0 \leqq \alpha < 1)$ is the Fourier transform of a distribution function which fails to be analytic at $x = 0$.[3]

In this note we discuss by means of examples just how much $\sigma(x)$ is restricted by the behavior of $\Lambda(u;\sigma)$ as $|\,u\,| \to \infty$. Our results may be stated descriptively as follows. No condition weaker than (1.3) ensures that $\sigma(x)$ has an interval of analyticity. Let $\eta(u)$ be any function which, when $|\,u\,| \to \infty$, decreases to zero more slowly than every function $e^{-c|u|}$ $(c > 0)$. Then there exists a distribution function $\sigma(x)$ whose points of non-analyticity form an arbitrary closed set (which may in particular be the empty set or the whole real axis), and whose Fourier transform $\Lambda(u;\sigma)$ is of the same order of magnitude as $\eta(u)$ when $|\,u\,| \to \infty$. Furthermore, there is a distribution function, belonging to a general Denjoy-Carleman quasi-analytic class on $(-\infty, \infty)$, and analytic except on an arbitrary closed set.

It is not known whether (1.3) is a necessary condition for $\sigma(z)$ to be analytic and bounded in the strip $|\,y\,| < c$. It may therefore be of interest that if $\sigma(z)$ is analytic and bounded in a portion of the complex plane somewhat more extensive than a strip, we can obtain an estimate for $\Lambda(u;\sigma)$ of the same type as (1.3). We shall, in fact, show that if $\sigma(z)$ is analytic and bounded in the "double funnel" $R = R_1 + R_2 + R_3$, where

$$\left.\begin{aligned} R_1 &= E_z[\,|\,y\,| < c], \\ R_2 &= E_z[\,|\,y\,| < |\,x - a\,|^{\gamma},\ x \geqq a], \\ R_3 &= E_z[\,|\,y\,| < |\,x + a\,|^{\gamma},\ x \leqq -a], \end{aligned}\right\}$$

and $c > 0$, $\gamma > 0$, $a \geqq 0$, then

$$\Lambda(u;\sigma) = O(|\,u\,|^{3/2} e^{-c|u|}) \qquad (|\,u\,| \to \infty). \tag{1.4}$$

The function $\sigma(z) = \frac{1}{2} + \frac{1}{\pi}\arctan z$ is analytic except at $z = \pm i$, $\sigma(x)$ is a distribution function, and $\Lambda(u;\sigma) = e^{-|u|}$; consequently we cannot expect to obtain an estimate much stronger than (1.4).

2. THEOREM 1. *Let $\epsilon(u)$ be an even function of u such that $\epsilon(u) > 0$ for all u, $\epsilon(u) \downarrow 0$ as $u \to \infty$, $u\epsilon(u) \uparrow \infty$ as $u \to \infty$ in $u \geqq u_0 > 0$, and $e^{-u\epsilon(u)}$ is convex in $u \geqq u_0$. Then there exists a distribution function $\tau(y)$, such that $y = 0$ is a point of non-analyticity of $\tau(y)$, and*

$$\Lambda(u;\tau) = O(e^{-|u|\epsilon(u)}) \qquad (|\,u\,| \to \infty). \tag{2.1}$$

The result of Theorem 1 will be made more precise in § 4, but the construction used here possesses independent interest.

[3] A. Wintner, "On a class of Fourier transforms," *American Journal of Mathematics*, vol. 58 (1936), pp. 45-90, 65.

We define an even function $L(u)$ by the relation

$$L(u) = \begin{cases} e^{-u\epsilon(u)} & (u \geqq u_0), \\ A + Bu & (0 \leqq u \leqq u_0), \end{cases}$$

where the constants A and B are chosen in such a way that $L(u)$ is convex and decreasing in $u > 0$. Set

$$\sigma(y) = \frac{1}{\pi}\int_0^\infty \frac{\sin uy}{u} L(u)\,du \qquad (-\infty < y < \infty),$$

where $\int_0^\infty$ is to be interpreted, if necessary, as $\lim_{T\to\infty}\int_0^T$. Then

$$\tau(y) = A^{-1}\sigma(y) + \tfrac{1}{2}$$

is a distribution function with the desired properties.

Clearly,

$$\Lambda(u;\tau) = A^{-1}L(|u|),$$

so that (2.1) is satisfied.

Since $L(u)$ is convex and $L(u) \downarrow 0$ as $u \to \infty$,

$$\sigma'(y) = \frac{1}{\pi}\int_0^\infty \cos uy\, L(u)\,du \geqq 0 \qquad (y \neq 0)^4$$

and $\sigma(y) > 0$ when $y > 0$. Hence $\sigma(y)$, and therefore $\tau(y)$, are non-decreasing.

We have

$$\sigma(y) = \frac{A}{\pi}\int_0^{u_0} \frac{\sin uy}{u}\,du + \frac{B}{\pi}\int_0^{u_0} \sin uy\,du + \frac{1}{\pi}\int_{u_0}^\infty L(u)\sin uy\,du$$
$$= I_1 + I_2 + I_3.$$

When $y \to \infty$, $I_1 \to \frac{1}{2}A$; $I_2 \to 0$; and, by the second mean-value theorem,

$$|I_3| \leqq 2L(u_0)/(\pi y) \to 0.$$

Hence $\sigma(y) \to \frac{1}{2}A$ as $y \to \infty$; similarly, $\sigma(y) \to -\frac{1}{2}A$ as $y \to -\infty$. Consequently $\tau(y)$ is a distribution function.

Set

$$f(z) = \frac{1}{\pi}\int_0^\infty e^{-uz}L(u)\,du \qquad (z = x + iy).$$

The abscissa of convergence of this Laplace integral is clearly $x = 0$. Since $L(u)$ is a positive function, $f(z)$ has a singularity at $z = 0$.[5] It follows that

$$\sigma'(y) = \Re[f(iy)]$$

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[4] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, 1937, p. 170.

[5] E. Landau, "Über einen Satz von Tschebyschef," *Mathematische Annalen*, vol. 61 (1905), pp. 527-550, 548.

is not analytic at $y=0$. For, if $\sigma'(y)$ is analytic at $y=0$, then $\sigma'(y)=g(iy)$, where $g(z)$ is analytic in a neighborhood of $z=0$; $f(z)-g(z)$ is purely imaginary on $x=0$, and consequently, by the Schwarz reflection principle, has an analytic extension in a neighborhood of $z=0$. Hence $f(z)$ cannot have a singularity at $z=0$, a contradiction.

3. LEMMA 1. *Let $\lambda(t)$ be a positive function defined for $t>0$, such that $\lambda(t)\downarrow 0$ as $t\to\infty$. Let $\mu(u)$ be the inverse function of $t/\lambda(t)$. Let $\kappa(n)$ be a real-valued function defined for $n=1,2,\cdots$. Let F be the closure of the set of points $\{\kappa(n)+i\lambda(n)\}$, and let E be the intersection of F with the real axis. Then there exists a distribution function $\sigma(x)$ such that E is the set of points of non-analyticity of $\sigma(x)$ and such that*

$$\Lambda(u;\sigma)=O(e^{-(1-\epsilon)\mu(|u|)}) \qquad (|u|\to\infty) \tag{3.1}$$

for any $\epsilon>0$.

Set

$$\tau(z)=\tfrac{1}{2}+\tfrac{1}{\pi}\operatorname{arc\,tan} z \qquad (z=x+iy).$$

We cut the z-plane along the imaginary axis from i to $i\infty$ and from $-i$ to $-i\infty$. We choose the one-valued branch of $\tau(z)$ which is regular in the cut plane and which is determined by the condition

$$\tau(x)\to 1 \qquad (x\to\infty).$$

Then $\tau(x)$ is a distribution function; and $\tau(z)$ is bounded on any bounded closed set in the cut plane. Set

$$\sigma(z)=(e-1)\sum_{n=1}^{\infty}e^{-n}\,\tau\left\{\frac{z-\kappa(n)}{\lambda(n)}\right\}. \tag{3.2}$$

We cut the z-plane along the lines $x=\kappa(n)$ from $\kappa(n)+i\lambda(n)$ to $\kappa(n)+i\infty$ and from $\kappa(n)-i\lambda(n)$ to $\kappa(n)-i\infty$, and denote the closure of the sum of these cuts by H. The series in (3.2) is uniformly convergent on any bounded set at a positive distance from the set H, and also uniformly convergent on the real axis. $\sigma(x)$ is a distribution function, and E is the set of points of non-analyticity of $\sigma(x)$.

We have

$$\Lambda(u;\tau)=e^{-|u|};$$

hence

$$\Lambda\left[u;\tau\left\{\frac{z-\kappa(n)}{\lambda(n)}\right\}\right]=e^{iu\kappa(n)-|u|\lambda(n)};$$

consequently

$$\Lambda(u;\sigma)=(e-1)\sum_{n=1}^{\infty}e^{-n+iu\kappa(n)-|u|\lambda(n)}. \tag{3.3}$$

It follows that

$$(e-1)^{-1}\,|\,\Lambda(u;\sigma)\,| \leqq \sum_{n=1}^{\infty} e^{-n-|u|\lambda(n)} = \sum_{n=1}^{v} + \sum_{n=v+1}^{\infty}$$
$$\leqq ve^{-|u|\lambda(v)} + \sum_{n=v+1}^{\infty} e^{-n} \qquad (v = 1, 2, \cdots).$$

Since $t/\lambda(t)$ increases, we have $|\,u\,|\,\lambda(v) \geqq v$ whenever $v \leqq \mu(|\,u\,|)$; we take $v = [\mu(|\,u\,|)]$. Then

$$(e-1)^{-1}\,|\,\Lambda(u;\sigma)\,| \leqq ve^{-v} + e^{-v}$$
$$= O(e^{-(1-\epsilon)\mu(|u|)}) \qquad (|\,u\,| \to \infty)$$

for every $\epsilon > 0$.

LEMMA 2. *Let $\lambda(t)$ satisfy the hypotheses of Lemma 1. If $\kappa(n) = b\xi(n)$ $(n = 1, 2, \cdots)$ where b is a constant and $\xi(n)$ is an integer-valued function, and $\sigma(z)$ is defined by (3.2), then*

$$\limsup_{|u|\to\infty}\,|\,\Lambda(u;\sigma)\,|\,e^{2\mu(|u|)} > 0. \tag{3.4}$$

By (3.3), if $bu = 2\pi h$, where h is an integer, we have

$$(e-1)^{-1}\Lambda(u;\sigma) = \sum_{n=1}^{\infty} e^{-n-|u|\lambda(n)} \geqq \sum_{n=v+1}^{\infty} e^{-n-|u|\lambda(n)},$$

where $v = [\mu(|\,u\,|)]$. When $n \geqq v+1$, $|\,u\,|\,\lambda(n) \leqq n$; hence

$$(e-1)^{-1}\Lambda(u;\sigma) \geqq \sum_{n=v+1}^{\infty} e^{-2n} = (e^2-1)^{-1}e^{-2[\mu(|u|)]}. \tag{3.5}$$

The result follows.

4. THEOREM 2. *Let $\delta(t)$ be a positive function defined for $t > 0$, such that $\delta(t) \downarrow 0$ and $t\delta(t) \uparrow \infty$ as $t \to \infty$. Then there exists a distribution function $\sigma(x)$, analytic on $-\infty < x < \infty$, such that*

$$\limsup_{|u|\to\infty}\,|\,\Lambda(u;\sigma)\,|\,e^{2|u|\delta(|u|)} > 0, \tag{4.1}$$

and

$$\limsup_{|u|\to\infty}\,|\,\Lambda(u;\sigma)\,|\,e^{(1-\epsilon)|u|\delta(|u|)} < \infty \tag{4.2}$$

for every $\epsilon > 0$.

We observe that $e^{t\delta(t)}$ can be made to approach infinity as slowly as we please when $t \to \infty$.

We set $\mu(u) = u\delta(u)$, and

$$\lambda(t) = \frac{t}{\mu^{-1}(t)} = \frac{\mu(\mu^{-1}(t))}{\mu^{-1}(t)} = \delta(\mu^{-1}(t)).$$

It is clear that $\lambda(t) \downarrow 0$ as $t \to \infty$. The functions $\lambda(t)$ and $\mu(u)$ satisfy the conditions of Lemma 1. We take $\kappa(n) = n$ $(n = 1, 2, \cdots)$. The distribution function $\sigma(x)$ defined by (3. 2) has, by Lemmas 1 and 2, the required properties.

THEOREM 3. *If $\delta(t)$ satisfies the conditions of Theorem 2, there exists a distribution function $\sigma(x)$, having $x = 0$ as its only point of non-analyticity, and such that $\Lambda(u;\sigma)$ satisfies* (4. 1), (4. 2).

The proof is parallel to that of Theorem 2, except that we take $\kappa(n) = 0$ $(n = 1, 2, \cdots)$. We note that $\sigma(x)$ has an extension which is analytic in each of the half-planes $\Re(z) > 0$, $\Re(z) < 0$.

THEOREM 4. *Let E be an arbitrary closed point set on the real axis. If $\delta(t)$ satisfies the conditions of Theorem 2, there exists a distribution function $\sigma(x)$, having E as its set of points of non-analyticity, and such that $\Lambda(u;\sigma)$ satisfies* (4. 1), (4. 2).

Let $\{\beta_n\}$ $(n = 1, 2, \cdots)$ be a sequence of points dense in E; let $\{p_n\}$ be a sequence of positive numbers whose sum is unity. Write $\rho(x)$ for the distribution function constructed in Theorem 3. Set

$$\sigma(z) = \sum_{n=1}^{\infty} p_n \rho(z - \beta_n). \tag{4. 3}$$

Then $\sigma(x)$ is a distribution function; it is analytic on the complement of E because the series in (4. 3) is uniformly convergent on any bounded point set at a positive distance from the lines $x = \beta_n$ $(n = 1, 2, \cdots)$; every point of E is a limit point of singularities of $\sigma(z)$, and hence a point of non-analyticity of $\sigma(x)$.

The Fourier transform of $\sigma(x)$ is

$$\Lambda(u;\sigma) = \Lambda(u;\rho) \sum_{n=1}^{\infty} p_n e^{iu\beta_n} = \Lambda(u;\rho)\Omega(u),$$

where $\Omega(u)$ is a uniformly almost periodic function. Since $\Lambda(u;\rho)$ satisfies (4. 2), and $\Omega(u)$ is bounded, $\Lambda(u;\sigma)$ satisfies (4. 2). Since for $\rho(x)$ the constant b of Lemma 2 vanishes, $\Lambda(u;\rho)$ satisfies (3. 5) for every u. Therefore, since $\Omega(u)$ is uniformly almost periodic, $\Lambda(u;\sigma)$ satisfies (4. 1).

5. Let $\epsilon(u)$ be a positive non-increasing function of u, defined in $(0, \infty)$, such that $u^{\frac{1}{2}}\epsilon(u)$ is non-decreasing for all sufficiently large u, and

$$\int^{\infty} \frac{\epsilon(u)}{u}\, du$$

is divergent. If $f(x) \in C^\infty(a, b)$, and

$$| f^{(k+1)}(x) |^{1/k} \leqq Ak/\epsilon(k) \qquad (a \leqq x \leqq b;\ k = 0, 1, 2, \cdots),$$

where A is independent of k and x, we shall say that $f(x)$ belongs to the class $Q\{\epsilon(u); a, b\}$. The classes $Q\{\epsilon(u); a, b\}$ are rather general Denjoy-Carleman quasi-analytic classes; they include, in particular, all the Denjoy classes, for which (for sufficiently large u)

$$\epsilon(u) = \frac{1}{\log u}, \qquad \epsilon(u) = \frac{1}{\log u \log\log u}, \cdots.$$

The class $Q\{1; a, b\}$ is the class of functions analytic on $a \leqq x \leqq b$.

It follows from a result of Ingham's [6] that if $\sigma(x)$ is a distribution function, and

$$\Lambda(u; \sigma) = O(e^{-c|u|\epsilon(|u|)}) \qquad (| u | \to \infty),$$

then $\sigma(x) \in Q\{\epsilon(u); a, b\}$ for any finite interval (a, b).

THEOREM 5. *Suppose that $\epsilon(u)$ satisfies the conditions stated above, and that $\epsilon(u) \downarrow 0$ as $u \to \infty$. Let E be an arbitrary closed point set on the real axis. Then there exists a distribution function $\sigma(x)$ such that $\sigma(x) \in Q\{\epsilon(u); a, b\}$ for every finite interval (a, b), and E is the set of points of non-analyticity of $\sigma(x)$.*

This is an immediate consequence of Theorem 4.

6. THEOREM 6. *Let $c > 0$, $\gamma > 0$, $a \geqq 0$. Let R be the open region $R_1 + R_2 + R_3$ of the z-plane $(z = x + iy)$, where*

$$\left.\begin{aligned} R_1 &= E_z[| y | < c], \\ R_2 &= E_z[| y | < | x - a |^\gamma,\ x \geqq a], \\ R_3 &= E_z[| y | < | x + a |^\gamma,\ x \leqq -a]. \end{aligned}\right\}$$

If $\sigma(z)$ is analytic and bounded in R, and $\sigma(x)$ is a distribution function,[7] *then*

(6.1) $$\Lambda(u; \sigma) = O(| u |^{3/2} e^{-c|u|}) \qquad (| u | \to \infty).$$

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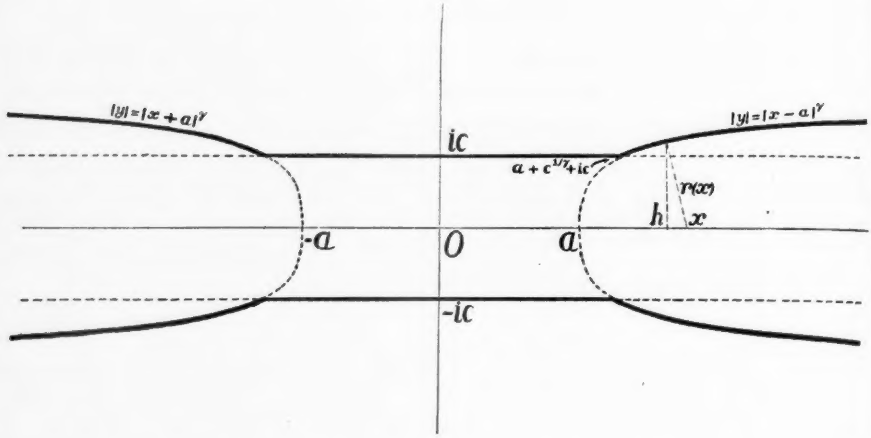
[6] A. E. Ingham, "A note on Fourier transforms," *Journal of the London Mathematical Society*, vol. 9 (1934), pp. 29-32.

[7] We do not use the full force of the hypothesis that $\sigma(x)$ is a distribution function. The theorem remains valid if we assume instead that $\sigma'(x) \in L(-\infty, \infty)$, and define the function $\Lambda(u; \sigma)$ by (1.1).



Let $r(x)$ denote the distance from the point x of the real axis to the boundary of R. Then

$$(6.2) \qquad |\sigma^{(n)}(x)| \leqq Mn!\, \{r(x)\}^{-n} \qquad (n = 0, 1, 2, \cdots),$$

where M is a bound for $|\sigma(z)|$ in R. We have

$$(6.3) \qquad r(x) \geqq c \qquad (-\infty < x < \infty).$$

Without loss of generality we may assume that $\gamma < \frac{1}{2}$. If $|x| > a$, it is obvious that, for some h such that $a < h < |x|$,

$$\{r(x)\}^{1/\gamma} = \{(|x| - h)^2 + (h - a)^{2\gamma}\}^{1/(2\gamma)}.$$

Hence, by Jensen's inequality,

$$\begin{aligned}\{r(x)\}^{1/\gamma} &\geqq (|x| - h)^{1/\gamma} + (h - a) \\ &= \lambda_1 + \lambda_2,\end{aligned}$$

say. When $|x| - h > 1$, $\lambda_1 > x - h$, so that $\lambda_1 + \lambda_2 > |x| - h + h - a = |x| - a > |x| - a - 1$. When $|x| - h \leqq 1$, we have $h \geqq |x| - 1$, so that $\lambda_2 = h - a \geqq |x| - a - 1$. Thus, in any case,

$$(6.4) \qquad r(x) \geqq (|x| - a - 1)^\gamma \qquad (|x| \geqq a + 1).$$

From (1.1) we obtain

$$\Lambda(u; \sigma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \sigma'(x)\, dx,$$

and hence, by integration by parts,

$$\Lambda(u; \sigma) = \frac{(-i)^k}{2\pi u^k} \int_{-\infty}^{\infty} e^{-iux} \sigma^{(k+1)}(x)\, dx \qquad (k = 0, 1, 2, \cdots);$$

the integrated terms vanish by (6.2) and (6.4). Consequently

$$(6.5)\qquad \begin{aligned} 2\pi \mid u \mid^k \mid \Lambda(u;\sigma) \mid &\leqq \int_{-\infty}^{\infty} \mid \sigma^{(k+1)}(x) \mid dx \\ &= \int_{-\infty}^{-b} + \int_{-b}^{b} + \int_{b}^{\infty} \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say, where $b = c^{1/\gamma} + a + 1$. By (6.2) and (6.3),

$$(6.6)\qquad I_2 \leqq 2M(k+1)!\, c^{-k-1}(c^{1/\gamma} + a + 1) \qquad (k = 0, 1, 2, \cdots).$$

By (6.2) and (6.4), for $k > 1/\gamma$,

$$(6.7)\qquad \begin{aligned} I_1 + I_3 &\leqq 2M(k+1)! \int_b^{\infty} (x - a - 1)^{-\gamma(k+1)} dx \\ &= \frac{2M(k+1)!}{\gamma(k+1) - 1}\, c^{-k-1+1/\gamma}. \end{aligned}$$

By (6.6) and (6.7),

$$I_1 + I_2 + I_3 \leqq A(k+1)!\, c^{-k-1} \qquad (k > 1/\gamma)$$

for some constant A. Hence by (6.5) and Stirling's formula,

$$(6.8)\qquad \mid \Lambda(u;\sigma) \mid \leqq A \mid u \mid (k+1)^{\frac{1}{2}} \left(\frac{k+1}{ce \mid u \mid}\right)^{k+1} \qquad (k > 1/\gamma)$$

with a new A.

For any u such that $\mid u \mid > (2\gamma + 1)/(\gamma c)$, we can take $k + 1 = [c \mid u \mid]$. It follows from (6.8) that

$$\mid \Lambda(u;\sigma) \mid \leqq A \mid u \mid^{3/2} e^{-c\mid u \mid} \qquad (\mid u \mid > (2\gamma + 1)/(\gamma c)),$$

with a new A, i. e.

$$\Lambda(u;\sigma) = O(\mid u \mid^{3/2} e^{-c\mid u\mid}) \qquad (\mid u \mid \to \infty).$$

This completes the proof.

PRINCETON UNIVERSITY AND
THE INSTITUTE FOR ADVANCED STUDY.

AN EXTENSION OF WIENER'S GENERAL TAUBERIAN THEOREM.*

By H. R. Pitt.

Suppose that $f(x)$ is defined in $(-\infty, \infty)$ and that two metrics $M\{f\}$, $M'\{f\}$ (which may be the same) satisfy the following postulates.

(a) $$M\{f\} = M\{|f|\}, \quad M'\{f\} = M'\{|f|\}.$$

(b) $$0 \leq M\{f\} \leq M'\{f\}.$$

(c) $$M\{f_1 + f_2\} \leq M\{f_1\} + M\{f_2\}, \quad M'\{f_1 + f_2\} \leq M'\{f_1\} + M'\{f_2\}.$$

(d) $$M_x\{f(x-y)\} = M\{f\}, \quad M'_x\{f(x-y)\} = M'\{f\}.$$

(e) If $k(x)$ belongs to $L(-\infty, \infty)$ and $M'_x\{f(x,y)\} \leq C$ for all values of y, then $g(x) = \int k(y) f(x,y)\, dy$ exists for all values of x and

$$M\{g\} \leq \int |k(y)|\, M_x\{f(x,y)\} dy,$$
$$M'\{g\} \leq \int |k(y)|\, M'_x\{f(x,y)\} dy.$$

(Integrals in which limits are not specified are from $-\infty$ to ∞.)

Examples of pairs of metrics which satisfy these postulates are the following.

(A) $$M'\{f\} = \overline{bd}\, |f(x)|, \quad M\{f\} = \overline{\lim_{x\to\infty}}\, |f(x)|.$$

(B) $$M'\{f\} = M\{f\} = \overline{bd}\, |f(x)|.$$

(C) $$M'\{f\} = \overline{bd}\left[\int_x^{x+1} |f(y)|^p dy\right]^{1/p}, \quad M\{f\} = \overline{\lim_{x\to\infty}}\left[\int_x^{x+1} |f(y)|^p dy\right]^{1/p} \quad (p \geq 1).$$

(D) $$M'\{f\} = \overline{bd}\left[\int_x^{x+1} |f(y)|^p dy\right]^{1/p}, \quad M\{f\} = \overline{\lim_{x\to\infty}}\left[\frac{1}{2x}\int_{-x}^{x} |f(y)|^p dy\right]^{1/p}.$$

(E) $$M'\{f\} = \overline{bd}\left[\int_x^{x+1} |f(y)|^p dy\right]^{1/p},$$
$$M\{f\} = \lim_{a\to\infty}\ \overline{bd}_{-\infty<x<\infty}\left[\frac{1}{a}\int_x^{x+a} |f(y)|^p dy\right]^{1/p}.$$

We shall prove the following theorem which includes Wiener's theorem when the metrics are defined as in (A).[1]

* Received May 17, 1938.

[1] H. R. Pitt, "General Tauberian theorems," *Proceedings of the London Mathematical Society*, vol. 43 (1938), Theorem 9, except that condition (4) is somewhat stronger than the condition $f(x) \epsilon T$.

THEOREM.

HYPOTHESIS. (1) $k(x)$ *belongs to* $L(-\infty,\infty)$,

(2) $K(x)=\int k(y)e^{-iyx}dy\neq 0 \qquad (-\infty<x<\infty)$,

(3) $M'\{f\}<\infty$,

(4) $M\{f(x+\epsilon)-f(x)\}=\delta(\epsilon), \qquad \lim_{\epsilon\to 0}\delta(\epsilon)=0$,

(5) $g(x)=\int k(x-y)f(y)dy$.

CONCLUSION. *If* $C_1>0$, *we can choose* C_2, *depending only on* C_1, $k(x)$ *and* $\delta(\epsilon)$, *so that*

$$M\{f\}\leq C_1+C_2M\{g\}.$$

Let $\lambda>0$,

$$h_\lambda(x)=\frac{\sin^2\lambda x}{\pi\lambda x^2}.$$

Using a theorem [2] on Fourier transforms due essentially to Wiener, we can write

$$h_\lambda(y)=\int k(y-t)p_\lambda(t)dt,$$

where $p_\lambda(t)$ belongs to $L(-\infty,\infty)$. Now

$$G(x)=\int|k(x-y)f(y)|\,dy$$

exists for all values of x, by (1), (3) and postulates (d) and (e), and

$$M\{G\}\leq M\{f\}\int|k(x)|\,dx.$$

Hence

$$\int|p_\lambda(t)|\,dt\int|k(x-t)f(x-y)|\,dy=\int|p_\lambda(t)G(x-t)|\,dt$$

exists for all values of x, and it follows that

$$\begin{aligned}\int p_\lambda(t)g(x-t)dt&=\int p_\lambda(t)dt\int k(y-t)f(x-y)dy\\&=\int f(x-y)dy\int k(y-t)p_\lambda(t)dt\\&=\int h_\lambda(y)f(x-y)dy.\end{aligned}$$

If we write $P(\lambda)=\int|p_\lambda(y)|\,dy$, it follows from (d) and (e) that

$$(6)\qquad M\{\int h_\lambda(y)f(x-y)dy\}\leq M\{g\}\int|p_\lambda(y)|\,dy=M\{g\}P(\lambda).$$

[2] *Ibid.*, Theorem 6. When the paper was written I was not aware that Professor Wiener had given the same theorem in his course of lectures at Massachusetts Institute of Technology.

Since $\int h_\lambda(y)\,dy = 1$, we can write

$$M\{f\} = M\{\int h_\lambda(y) f(x-y)\,dy + \int h_\lambda(y)[f(x) - f(x-y)]\,dy\},$$

and using (4), (6) and (e), we have

$$\begin{aligned} M\{f\} &\leq P(\lambda) M\{g\} + \int h_\lambda(y) M\{f(x) - f(x-y)\}\,dy \\ &= P(\lambda) M\{g\} + \int h_\lambda(y)\delta(y)\,dy \\ &= P(\lambda) M\{g\} + \int h(y)\delta(y/\lambda)\,dy. \qquad (7) \end{aligned}$$

Since $\delta(y/\lambda) \leq 2M'\{f\}$, by (b) and (c), and $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$, we can choose $\lambda = \lambda(C_1, \delta(\epsilon))$ so that

$$\int h(y)\delta(y/\lambda)\,dy \leq C_1,$$

and then (7) may be written

$$M\{f\} \leq C_1 + C_2 M\{g\},$$

where $C_2 = P(\lambda)$ depends only on C_1, $k(x)$ and $\delta(\epsilon)$.

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SYSTEMS OF DIFFERENTIAL EQUATIONS. I. THEORY OF IDEALS.*

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In publications of ourselves and of Raudenbush,[1] there was presented a theory of systems of differential equations which are algebraic in the unknowns and their derivatives. It is our object to start, with the present paper, the consideration of more general systems. We shall begin by treating systems of equations which are *analytic* in the independent variable and in the unknowns, and *algebraic* in the derivatives of the unknowns. Such a theory can be used as the basis for the study of any *finite* system of analytic differential equations, since, by replacing derivatives by new unknowns, the finite system can be made equivalent to a system involving only derivatives of the first order, and those algebraically.

This paper presents, for systems with the generality described above, a theory of ideals of the type given by Raudenbush for algebraic differential systems. In addition to the methods of our own earlier publications, we use the ideas in the very fundamental work of Raudenbush and also the methods of Rückert's important paper *Zum Eliminationsproblem der Potenzreihenideale.*[2] The present paper is complete in itself, except for the employment of Späth's extension of the Weierstrass preparation theorem. The use of Späth's theorem was suggested by Rückert's work. For unity of exposition, we have included a treatment of some questions which are covered by the work of Raudenbush. References to Raudenbush will indicate where we have borrowed from him.

Elements.

1. We shall study expressions in the unknown functions $y_1, \cdots, y_n$ of the independent variable x. We denote the j-th derivative of y_i by y_{ij}. We write, frequently, $y_i = y_{i0}$.

2. We consider any finite set of the y_{ij} with $j > 0$. Let a finite number

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[1] Ritt, "Differential equations from the algebraic standpoint," *Colloquium Publications of the American Mathematical Society*, vol. 14 (New York, 1932); Raudenbush, "Ideal theory and algebraic differential equations," *Transactions of the American Mathematical Society*, vol. 36 (1934), pp. 361-368.

[2] *Mathematische Annalen*, vol. 107 (1933), p. 259.

of power products in these y_{ij} be formed and let each power product be multiplied by a power series in $x; y_1, \cdots, y_n$ which converges for numerically small values of those letters. The sum of the terms thus formed will be called an *element*. In other words, an element is a polynomial in any set of the derivatives (proper)[3] of the y_i, with coefficients which are functions of $x; y_1, \cdots, y_n$ analytic at $(0; 0, \cdots, 0)$.

Elements will be denoted by capital italic letters and systems of elements by large Greek letters.

We denote by Γ the totality of elements which are free of the independent variable x. Thus an element G is in Γ if G is a polynomial in certain y_{ij} with $j > 0$, with coefficients which are functions of $y_1, \cdots, y_n$ analytic at $(0, \cdots, 0)$.

Ordering of letters and elements.

3. We order the letters y_{ij} as follows. We shall call y_{ij} *higher* than y_{kl} if $j > l$ or if $j = l$ and $i > k$.

Let A be an element involving one more y_{ij} effectively. The highest letter y_{ij} in A will be called the *leader* of A.

4. We consider any element A and any definite y_{ij}. If A, written as a series of powers of y_{ij}, contains only a finite number of terms, we shall, as is natural, call A a *polynomial in* y_{ij}. Of course every element is a polynomial in every y_{ij} with $j > 0$.

5. Let A be a non-constant element in Γ (§ 2) and B any element in Γ.[4] If B is a constant, or if B is not a constant and has a leader which is lower than the leader of A, we shall say that A *is of higher rank than* B.

Now let A and B, in Γ and neither a constant, have the same leader y_{jk} and let A and B be polynomials in y_{jk}. If A is of higher degree than B in y_{jk}, we shall say, again, that A is of higher rank than B. If A and B have the same degree in y_{jk}, we shall say that A and B are of the same rank.

Any two constants will be considered to be of equal rank.

Thus, the only pairs A, B in Γ whose two elements we do not compare, are pairs in which A and B are not constants, and have the same leader, with at least one of A and B failing to be a polynomial in the common leader.

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[3] Each y_i will be considered as its own derivative of order zero. Proper derivatives are derivatives of positive order.

[4] For our present purposes, only elements in Γ require ordering. The definition which follows immediately has somewhat more generality than is necessary for the applications.

Ascending sets.

6. Let A be a non-constant element in Γ which is a polynomial in its leader y_{jk}. An element B in Γ will be said to be *reduced with respect to A* if either

(a) $B = 0$

or

(b) *B is distinct from zero, contains no* y_{jl} *with*[5] $l > k$ *and is a poly-*

7. A set of non-constant elements in Γ

$$A_1, \cdots, A_p \tag{1}$$

with $p > 1$ will be called an *ascending set* if every A_i is a polynomial in its leader and if, for $j > i$, A_j is higher than A_i and reduced with respect to A_i.

An element A_1 in Γ will be called an ascending set if A_1 is non-constant and is a polynomial in its leader, or if A_1 is a constant distinct from zero.

Let (1) be an ascending set with $p > 1$. Let A_i and A_j be two elements of (1) with $j > i$. Let y_{ab} be the leader of A_i and y_{cd} that of A_j. Because A_j is higher than A_i and is reduced with respect to A_i, y_{cd} must be higher than y_{ab}. It follows that a and c are distinct. Thus *the p leaders* y_{ij} *of the elements in* (1) *have distinct i.* This implies that $p \leqq n$.

The ascending set (1)[6] will be said to be *of higher rank* than the ascending set

$$B_1, \cdots, B_q$$

if either

(a) *there is a j, exceeding neither p nor q, such that* A_i *and* B_i *are of the same rank for* $i < j$ *and that* A_j *is higher than* B_j,[7] or

(b) $q > p$ *and* A_i *and* B_i *are of the same rank for* $i \leqq p$.

Two ascending sets for which no difference in rank is created by what precedes will be said to be of the same rank. For such sets $p = q$ and A_i and B_i are of the same rank for every i.

The above ordering of ascending sets is easily seen to be transitive.

8. We prove the following lemma:

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[5] The j in y_{jl} is that which appears in y_{jk}.

nomial in y_{jk} *with a degree in* y_{jk} *which is lower than that of A in* y_{jk}.

[6] Here, as in all later work, it is understood that p may be unity.

[7] If $j = 1$, this is to mean that A_1 is higher than B_1.

LEMMA. *Every finite or infinite aggregate of ascending sets contains an ascending set whose rank is not higher than that of any other ascending set in the aggregate.*

Let us consider the first elements of all of the sets of the aggregate. We shall show that there is one of these first elements which is no higher than any other first element. This is certainly true if one of the first elements is a constant. Otherwise, we select those first elements which have a lowest leader and, from the elements just selected, we choose one which is of a lowest degree in the common leader.

Let us consider now those ascending sets in the given aggregate whose first elements have a least rank. Let σ_1 be the totality of such ascending sets. If the sets in σ_1 all consist of one element, any set in σ_1 will serve as the set whose existence was to be proved. Suppose that σ_1 contains sets which consist of more than one element. From among all such sets in σ_1 we select those whose second elements have a least rank and denote the totality of the sets selected by σ_2. If there are sets in σ_2 consisting of more than two elements, we make a third selection. As no ascending set has more than n elements, we arrive eventually at a collection of ascending sets of least rank.

9. Let (1) be an ascending set with A_1 non-constant. An element G in Γ will be said to be *reduced with respect to* (1) if G is reduced with respect to A_i, $i = 1, \cdots, p$.

Basic sets.

10. Let Σ be any system of elements in Γ which either contains a non-constant element which is a polynomial in its leader or else contains a constant distinct from zero. Because elements of the two types just mentioned are ascending sets, Σ contains ascending sets. By § 8, certain of the ascending sets in Σ have a least rank. Any such ascending set of least rank will be called a *basic set* of Σ.

Let Σ be a system in Γ for which (1), with A_1 non-constant, is a basic set. Then *no element in Σ which is an ascending set can be reduced with respect to* (1). Suppose that such an element, F, exists. Then F must be higher than A_1, else F would be an ascending set lower than (1). Similarly, F must be higher than A_2, else A_1, F would be an ascending set lower than (1). Finally, F is higher than A_p. Then $A_1, \cdots, A_p, F$ is an ascending set lower than (1). This proves our statement.

Let Σ be as in the preceding paragraph. We see that *if an element which is an ascending set and is reduced with respect to* (1), *is adjoined to Σ, the basic sets of the resulting system are lower than* (1).

Reduction.

11. Let G be any non-constant element in Γ which is a polynomial in its leader y_{ij}. We shall call the element $\partial G/\partial y_{ij}$ the *separant* of G. The coefficient of the highest power of y_{ij}, in G considered as a polynomial in y_{ij}, will be called the *initial* of G. G is higher than its separant and higher than its initial.

12. Let (1) be any ascending set with A_1 non-constant. Let S_i and I_i denote the separant and initial respectively of A_i, $i = 1, \cdots, p$.

We shall prove the following result.

Let G be an element in Γ which is a polynomial in the leader of every A_i and in all derivatives of each such leader. There exist non-negative integers s_i, t_i, $i = 1, \cdots, p$, which are such that when a suitable linear combination of the A_i and of derivatives,[8] *of certain orders, of the A_i, with elements for coefficients, is subtracted from*

$$S_1^{s_1} \cdots S_p^{s_p} I_1^{t_1} \cdots I_p^{t_p} G,$$

the remainder, R, is reduced with respect to (1).

Let u_i represent the leader of A_i in (1), $i = 1, \cdots, p$. We limit ourselves, as we may, to the case in which G involves derivatives, proper or improper, of one or more u_i. Let the highest letter in G which is a derivative of a u_i be represented by v and let v be a derivative of u_j. There is, of course, only one possibility for j.

To fix our ideas, we assume v higher than u_p. Using the algorithm of division, we find a relation

$$S_j^q G = CA'_j + B$$

where A'_j is a derivative of A_j, of some order, with v for leader; and where B is free of v. Because A'_j and S_j involve no letter higher than v, B involves no u_i-derivative [9] which is as high as v. For uniqueness of procedure, we take q as small as possible. We note that B is a polynomial in the u_i-derivatives.[10]

If B involves a u_i-derivative which is higher than u_p, we give B the treatment accorded to G. After a finite number of steps, we arrive at a unique

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[8] By the j-th derivative of an element A, we mean the element which is obtained by differentiating A j times with respect to x, the y_i being regarded as functions of x.

[9] Thus we denote a derivative (proper or improper) of a u_i.

[10] If w is a letter in A'_j with respect to which A'_j is not a polynomial, then A_j cannot be a polynomial in w. Thus w is not a u_i-derivative.

element D which differs by a linear combination of derivatives of the A_i from an element

$$S_1^{g_1} \cdots S_p^{g_p} G$$

and involves no u_i-derivative higher than u_p. D is a polynomial in the u_i-derivatives.

We then find a relation

$$I_p^{t_p} D = HA_p + K$$

where K is reduced with respect to A_p. K may involve u_p. Aside from u_p, the u_i-derivatives present in K are derivatives of $u_1, \cdots, u_{p-1}$. Such derivatives present in K are lower than u_p. Let v_1 be the highest of them.

Suppose that v_1 is higher than u_{p-1}. We give K the treatment accorded to G. In a finite number of steps we arrive at a unique element L which differs from some

$$S_1^{r_1} \cdots S_{p-1}^{r_{p-1}} I_{p-1}^{t_{p-1}} K$$

by a linear combination of A_{p-1} and proper derivatives of $A_1, \cdots, A_{p-1}$. L is reduced with respect to A_{p-1}.

Let us show that L is also reduced with respect to A_p. The first operation performed on K gives a relation

$$S_j^h K = EA'_j + K_1. \tag{2}$$

The highest letter in A'_j is v_1 and v_1 is lower than u_p. The highest letter in S_j is not higher than u_{p-1}. Thus, K_1 cannot contain a proper derivative of u_p. Let the degrees of K and K_1 in u_p be d and d_1 respectively. Suppose that $d_1 > d$. As S_j and A'_j are free of u_p, E in (2) must be of degree d_1 in u_p. Then EA'_j contains a term involving v_1 which is of degree d_1 in u_p. Such a term can be balanced neither by K_1 nor by the first member of (2). Thus K_1 is reduced with respect to A_p. Similar observations apply to the later steps in the calculation of L, so that L is reduced with respect to A_p.

Continuing, we reach a unique R as described above. We call R the *remainder of G with respect to* (1).

Series.

13. An element in Γ will be called a *series* if the element involves no y_{ij} with $j > 0$. A series is simply a function of $y_1, \cdots, y_n$ analytic at $(0, \cdots, 0)$.

Let A be a series which is not a constant. By the *class* of A, we shall

mean the greatest j such that A involves y_j effectively. Constants will be called series of class zero.

Let A be a series of class $j > 0$. We shall call A *regular* if A contains a term ay_j^m with $m > 0$ and a a constant distinct from zero.

Regular systems.

14. Let Σ be a system of elements in Γ. We shall call Σ *regular* if either

(a) *A positive integer $j \leqq n$ exists such that Σ contains regular series of each of the classes $n, n-1, \cdots, j$ but contains no series of positive class less than j.*[11]

or

(b) *Σ contains no series of positive class.*

If (a) holds, we shall call $j-1$ the *index* of Σ. If (b) holds, the index of Σ will be defined as n.

15. Let Σ be any system of elements in Γ. We shall show that, by a non-singular linear transformation on $y_1, \cdots, y_n$, with constant coefficients, Σ can be converted into a regular system.[12]

If Σ contains no series of positive class, Σ is regular. Suppose that series of positive class exist in Σ and let A be one of them. By a non-singular linear transformation of the type

$$y_i = a_{i1}y'_1 + \cdots + a_{in}y'_n, \qquad (i = 1, \cdots, n),$$

with rational a_{ij}, we can convert A into a series B in the y'_i which is of class n and regular.[13] Under this transformation, Σ goes over into a system Σ' in the y'_i. If Σ' contains no series of one of the classes $1, \cdots, n-1$, Σ' is regular. Let Σ' contain a series C of positive class less than n. By a non-singular linear transformation applied to $y'_1, \cdots, y'_{n-1}$, we convert C into a regular series of class $n-1$. This second transformation converts B into a regular series of class n.

A sufficient number of repetitions of the above procedure furnishes a non-singular transformation which converts Σ into a regular system.

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[11] If $j = 1$, this is to mean that Σ contains regular series of the classes $n, n-1, \cdots, 1$.

[12] Cf. Rückert, *loc. cit.*, p. 266.

[13] It is only necessary to regularize, in the manner explained in treatises of algebra, any one of the homogeneous polynomials of which A is the sum.

Residues.

16. The following important theorem is due to Späth.[14] *Let A and B be power series in $u_1, \cdots, u_p$ with numerical coefficients, convergent for $|u_i| < r$, $i = 1, \cdots, p$, where $r > 0$. For each j, let α_j, a series in $u_1, \cdots, u_{p-1}$, be the coefficient of $u_p{}^j$ in A arranged as a power series in u_p. Let a positive integer q exist such that $\alpha_0, \alpha_1, \cdots, \alpha_{q-1}$ vanish for $u_1 = \cdots = u_{p-1} = 0$, while α_q does not. There exists one and only one relation*

$$(3) \qquad B = CA + D$$

where C and D are power series in $u_1, \cdots, u_p$ convergent for numerically small values of the u_i, and where no exponent of u_p in D is greater than $q-1$.

That is,

$$D = \beta_0 u_p{}^{q-1} + \cdots + \beta_{q-1}$$

with the β_i power series in $u_1, \cdots, u_{p-1}$.

17. Let now Σ be a regular system of index $s < n$ (§ 14). Let

$$(4) \qquad B_{s+1}, B_{s+2}, \cdots, B_n$$

be series in Σ with B_i of class i and regular, $i = s+1, \cdots, n$.

Let G be any element in Σ. If B_n does not vanish for $y_1 = \cdots = y_n = 0$, we have $G = M_n B_n$ with M_n an element. If the vanishing does occur, we arrange G as a polynomial in the y_{ij} with $j > 0$ and apply Späth's theorem to the coefficients in G. We find a relation

$$(5) \qquad G = M_n B_n + G_1$$

with G_1 a polynomial in y_n as well as in the y_{ij} with $j > 0$. Considering G_1 as such a polynomial, we give it, with respect to B_{n-1}, the treatment accorded to G with respect to B_n. Continuing, we reach a relation

$$G = M_n B_n + \cdots + M_{s+1} B_{s+1} + H$$

with the M elements and H an element which is a polynomial in y_i, $s+1 \leqq i \leqq n$. We shall call H the *residue of G with respect to* (4).

Bases.

18. Let Σ be any system of elements[15] and Φ a finite subset of Σ. We

[14] *Journal für die reine und angewandte Mathematik*, vol. 161 (1929), p. 95.

[15] Not necessarily in Γ.

shall call Φ a *basis* of Σ if, for every element G in Σ, a positive integer p depending on G exists such that G^p is a linear combination of the elements in Φ and of their derivatives (of arbitrary orders), with elements for coefficients.

§§ 19-23 are devoted to the proof of the lemma:

LEMMA. *Every infinite system of elements has a basis.*

We shall prove this lemma first for systems of elements in Γ. It will then be easy to pass to the general case. We note that if a system Σ of elements in Γ has a basis Φ, the expression of any G^p (as above) in terms of the elements of Φ may be assumed to have coefficients in Γ. In short, one may replace x, in the coefficients, by 0.

19. The notation

$$P \equiv Q \qquad (C_1, \cdots, C_p),$$

where all letters represent elements, will mean that $P - Q$ is a linear combination of $C_1, \cdots, C_p$ and their derivatives (of arbitrary orders), with elements for coefficients.[16]

The following lemma is due to Raudenbush:[17]

LEMMA. *If*

$$(PQ)^g \equiv 0 \qquad (C_1, \cdots, C_p) \tag{6}$$

where g is some positive integer, a positive integer h exists such that

$$(P'Q)^h \equiv 0 \qquad (C_1, \cdots, C_p), \tag{7}$$

where P' is the derivative of P.

The relation $P^gQ^g \equiv 0$ gives,[18] by differentiation,

$$gP^{g-1}P'Q^g + gP^gQ^{g-1}Q' \equiv 0.$$

Multiplying through by Q and dividing by g, we have

$$P^{g-1}P'Q^{g+1} \equiv 0. \tag{8}$$

If $g = 1$, we have $(P'Q)^2 \equiv 0$. Suppose that $g > 1$. We differentiate (8), multiply through by $P'Q$ and use (6) and (8). We find that

$$P^{g-2}P'^3Q^{g+2} \equiv 0.$$

If $g = 2$ we have (7) with $h = 4$. In general, (7) holds with $h = 2g$.

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[16] When the letters above represent elements in Γ, the coefficients can be taken in Γ.

[17] *Loc. cit.*, §§ 1, 2.

[18] All congruences are with respect to $C_1, \cdots, C_p$.

20. We now assume that there exist infinite systems of elements in Γ which have no bases. We work towards a contradiction.

LEMMA. *Let Σ be a system of elements which has no basis. Let $F_1, \cdots, F_p$ be elements such that when each element in Σ is multiplied by a suitable product of non-negative powers of $F_1, \cdots, F_p$, one obtains a system Λ which has a basis. Then the system*[19] *$\Sigma + F_1F_2 \cdots F_p$ has no basis.*[20]

Let $\Sigma + F_1 \cdots F_p$ have a basis. If, to a basis of a system, one adds any finite number of elements of the system, one secures a new basis for the system. Then let $\Sigma + F_1 \cdots F_p$ have a basis.

$$F_1 \cdots F_p;\ H_1, \cdots, H_r, \tag{9}$$

where the H_i are such that the elements of Λ which they yield, after the above described multiplications, form a basis for Λ. Let G be any element in Σ. Let K represent $F_1 \cdots F_p$. Because Λ has the basis mentioned above, we have, for some g,

$$(KG)^g \equiv 0 \qquad (H_1, \cdots, H_r). \tag{10}$$

We have, furthermore, for some h, referring to (9),

$$G^h = MK + M_1K' + \cdots + N_1H_1 + \cdots, \tag{11}$$

with K' the derivative of K, the unwritten terms involving either higher derivatives of K or a derivative, proper or improper, of an H_i.

Let t be a positive integer and let both sides of (11) be raised to the t-th power. If t is large, every term in the expression for G^{ht} which is free of the H_i and their derivatives will involve to a high power either K or some derivative of K appearing in (11). We multiply the expression for G^{ht} by a high power of G and then use (10) and § 19.[21] We see that a sufficiently high power of G is linear in the H_i and their derivatives. This means that $H_1, \cdots, H_r$ is a basis for Σ, so that the lemma is proved.

21. LEMMA. *Let Σ be a system of elements which has no basis. Let $F_1, \cdots, F_p$ be elements such that $\Sigma + F_1 \cdots F_p$ has a basis. Then at least one of the systems $\Sigma + F_i$, $i = 1, \cdots, p$, has no basis.*[20]

We may evidently limit ourselves to the case of $p = 2$. Let each of $\Sigma + F_1$, $\Sigma + F_2$ have a basis. Then we can find a set of elements $H_1, \cdots, H_r$

[19] The logical sum of Σ and $F_1 \cdots F_p$.

[20] Raudenbush, § 4.

[21] Actually, we are using a trivial extension of the result in § 19.

in Σ such that $F_1, H_1, \cdots, H_r$ is a basis for $\Sigma + F_1$ and $F_2, H_1, \cdots, H_r$ a basis for $\Sigma + F_2$. Let G be any element in Σ. Let g be such that G^g is linear in derivatives (proper or improper) of F_1 and the H_i and also linear in derivatives of F_2 and the H_i. If we multiply together these two expressions for G^g, we have G^{2g} expressed as a linear combination of certain $F_1^{(j)}F_2^{(k)}$, where superscripts denote differentiation, and of the H_i and their derivatives. Now, given any $F_1^{(j)}F_2^{(k)}$, a sufficiently high power of it is, by § 19, linear in F_1F_2 and its derivatives. We see now that a sufficiently high power of G^{2g} is linear in $F_1F_2, H_1, \cdots, H_r$ and their derivatives. Then $F_1F_2, H_1, \cdots, H_r$ is a basis for $\Sigma + F_1F_2$. This contradiction proves the lemma.

22. We now complete, for systems of elements in Γ, the proof of the lemma stated in § 18.

Let there exist a system in Γ which has no basis. By § 15, there will exist a regular system without a basis. Let s, with $0 \leqq s \leqq n$ be the least integer which is the index of a regular system without a basis.

We suppose first that $s < n$.

Let Σ be any regular system of index s which has no basis. We consider a set (4) contained in Σ. We observe that every B_i in (4) vanishes when its letters are replaced by zero. A B_i which did not so vanish would be a basis of Σ. By the Weierstrass preparation theorem, we have $B_i = A_iB'_i$, $i = s+1, \cdots, n$, with B'_i a regular series of class i which is a polynomial in y_i and A_i a series of class not exceeding i which does not vanish for zero values of its letters. When each B_i is replaced in Σ by B'_i, we obtain a regular system Σ_1. Σ_1 has no basis, for if it had a basis composed of the B'_i and other elements $H_1, \cdots, H_r$, Σ would have a basis composed of the B_i and of $H_1, \cdots, H_r$.

From each B'_j with $j > s+1$, we can subtract a linear combination of the B'_i with $i < j$, with series for coefficients, so as to obtain a regular series C_j of class j which is a polynomial in $y_{s+1}, \cdots, y_{j-1}$ as well as in y_j. Let Σ_2 result from Σ_1 by replacing each B'_j with $j > s+1$ by C_j. Then Σ_2 is a regular system of index s which has no basis. Let G be any element of Σ_2 which is not in

$$B'_{s+1}, C_{s+2}, \cdots, C_n. \tag{12}$$

Let H be the residue of G with respect to (12). Let Σ_3 be the system composed of (12) and of the totality of the H. Then Σ_3 has no basis. Σ_3 cannot contain a series of positive class less than $s+1$; if it did, it could be transformed into a regular system of index less than s, without a basis. Thus Σ_3 is regular and of index s.

There exist thus regular systems of index s which lack bases, in which each element of the system is a polynomial in every letter not lower than y_{s+1}. Of all such systems, let Σ be one whose basic sets are not higher than those of any other such system (§ 8). Let

$$(13) \qquad A_1, \cdots, A_p$$

be a basic set of Σ. Then A_1 is not a constant; if it were, it would be a basis of Σ. Let S_i and I_i be the separant and initial respectively of A_i, $i = 1, \cdots, p$. Let

$$(14) \qquad B_{s+1}, \cdots, B_n$$

be a set (4) in Σ.

Let G be any element of Σ which is not in (13) or (14). Let R be the remainder of G with respect to (13).

Let Ω be the system composed of (13), (14) and the totality of the R. We shall prove that Ω has a basis.

Let this be false. Then there are certainly non-zero R in Ω. If no R is a series of positive class less than $s + 1$, Ω is a regular system of index s, with each element a polynomial in every letter not lower than y_{s+1}, whose basic sets, by § 10, are lower than (13). This is incompatible with the absence of a basis. If some R is a series of positive class less than $s + 1$, we can transform Ω into a regular system of index less than s.

Thus Ω has a basis. Let Λ be the system composed of (13), (14) and of the products of the general type

$$(15) \qquad S_1^{s_1} \cdots I_p^{t_p} G$$

which are used in forming the R in Ω. Let a basis of Ω consist of (13), (14) and of certain R, say $R_1, \cdots, R_q$. Then Λ has a basis consisting of (13), (14) and the elements (15) which give $R_1, \cdots, R_q$.

The lemmas of §§ 20, 21 inform us now that at least one of the systems $\Sigma + S_i$, $\Sigma + I_i$ has no basis. If, in such a system without a basis, the S_i or I_i adjoined is not a series of positive class less than $s + 1$, the system is regular and has basic sets lower than (13). The case of a series of positive class less than $s + 1$ is handled by the usual transformation. The hypothesis that Σ has no basis is thus untenable in the case of $s < n$.

For $s = n$, we take Σ as a regular system, lacking a basis, whose basic sets are as low as possible. The proof proceeds as above, without the set (14).

23. Now let Σ be an infinite system of elements which are not necessarily in Γ. We shall prove that Σ has a basis. Let G be any element in Σ

In the expression for G in terms of x and the y_{ij}, we replace x by a new letter y_0. Then G goes over into an element G' in the functions $y_0, y_1, \cdots, y_n$ of the variable x. Σ goes over, in this manner, into a system Σ' in $y_0, \cdots, y_n$, Σ' being of the type examined in the preceding sections. Let $H'_1, \cdots, H'_r$ be a basis of Σ', H'_i being derived, as above, from an element H_i in Σ.

Referring to any G and G' as above, let G'^t be linear in the H'_i and their derivatives. We consider the equation which expresses G'^t in terms of the H'_i. In this equation, we replace y_0 by x, y_{01} by 1 and y_{0j} with $j > 1$ by 0. There results an expression for G^t in terms of the H_i and their derivatives. Thus, $H_1, \cdots, H_r$ is a basis for Σ.

Ideals.

24. A system Σ is called an *ideal* if it has the following properties:

(a) *Given any finite set of elements of Σ, every linear combination of those elements, with elements for coefficients, is in Σ.*

(b) *Given any element in Σ, the derivative of the element is also in Σ.*

An ideal Σ is called *perfect* [22] if, whenever an element is such that some power of it belongs to Σ, the element belongs to Σ. An ideal Σ is called *prime* if whenever AB is in Σ, at least one of A and B is in Σ. Every prime ideal is perfect.

We prove the following theorem: [22]

Every perfect ideal of elements is the intersection of a finite number of prime ideals.

Let Σ be a perfect ideal. Suppose that Σ is not the intersection of a finite number of prime ideals. Then Σ is not prime. Let AA', but neither A nor A', belong to Σ.

Let $H_1, \cdots, H_r$ be any basis of Σ. Let G be any element such that, for some g,

$$G^g \equiv 0 \qquad (H_1, \cdots, H_r, A). \tag{16}$$

The details of the preceding sections show that the totality Σ_1 of such elements G is a perfect ideal. Using A', in place of A, we obtain similarly a perfect ideal Σ'_1.

We prove now that Σ is the intersection of Σ_1 and Σ'_1. We note first that Σ is a proper subset of each of Σ_1 and Σ'_1. Now let G be any element common to Σ_1 and Σ'_1. We consider (16), taking g large enough to have also

[22] Raudenbush, § 3.

$$(17) \qquad G^g \equiv 0 \qquad (H_1, \cdots, H_r, A').$$

As in § 21, we see that, for some t,

$$G^{2gt} \equiv 0 \qquad (H_1, \cdots, H_r, AA'),$$

which means that G is in Σ.

At least one of Σ_1 and Σ'_1 is not the intersection of a finite number of prime ideals. Let this be the case for Σ_1. We find, as above, a perfect ideal Σ_2 of which Σ_1 is a proper part. Continuing, we find an infinite sequence of perfect ideals,

$$(18) \qquad \Sigma, \Sigma_1, \Sigma_2, \cdots$$

each a proper part of its successor. Let Ω be the logical sum of the ideals in (18). Let $H_1, \cdots, H_r$ be a basis of Ω. There is some Σ_p which contains all of the H_i. That Σ_p contains Ω. This contradiction proves that Σ is the intersection of a finite number of prime ideals.

Now, let the perfect ideal Σ be the intersection of the prime ideals

$$(19) \qquad \Sigma_1, \cdots, \Sigma_r.$$

Suppressing some of the Σ_i if necessary, we suppose that no Σ_i contains any Σ_j with $j \neq i$. We shall then prove that (19) is *unique*, that is, that if Σ is the intersection of prime ideals

$$\Omega_1, \cdots, \Omega_s$$

with no Ω_i containing any Ω_j with $j \neq i$, then $r = s$ and each Σ_i is identical with some Ω_j.

We show first that Σ_1 contains some Ω_j. Let this be false. Let A_i be an element in Ω_i, $i = 1, \cdots, s$, which is not in Σ_1. Then $A_1 A_2 \cdots A_s$, which is in every Ω_i, hence in Σ, is not in Σ_1. This is impossible. Let, then, Σ_1 contain Ω_1. Now, Ω_1, similarly, contains some Σ_j which must be Σ_1 since Σ_1 cannot contain a Σ_j with $j \neq 1$. The uniqueness is proved.

COLUMBIA UNIVERSITY.

DETERMINATION OF A VAN DER CORPUT ABSOLUTE CONSTANT.*

By Richard Kershner.

A very useful lemma of van der Corput [1] is to the effect that there exist absolute constants μ, ν with the property that if $f(x)$ be a real valued function possessing, in the finite interval $[a, b]$, a second derivative nowhere less than a fixed positive constant r, then

$$\left| \int_a^b \exp(if(x))\,dx \right| \leqq \mu r^{-1/2} \tag{1}$$

and

$$\left| \int_a^b \cos f(x)\,dx \right| \leqq \nu r^{-1/2} \tag{2}$$

it being understood that μ, ν are independent of $[a, b]$ as well as of $f(x)$.

The author has given [2] a very elementary proof of this lemma in a way that yields the best possible value ν_0 of ν satisfying (2). In fact it was shown that the expression on the left of (2) attains a maximum, under the restriction $f''(x) \geqq r > 0$, if the function $f(x)$ is chosen to be a parabola $f(x) = rx^2/2 + c_0$, for a certain constant c_0, and if $-a = b = [(\pi - 2c_0)/r]^{1/2}$. The constant c_0 is determined as the only root c of the equation

$$\int_0^{(\pi/2-c)^{1/2}} \sin(x^2 + c)\,dx = 0 \tag{3}$$

in $-\pi/2 \leqq c \leqq \pi/2$. This explicit determination of the maximum of the left side of (2) furnished the decimal approximation $\nu_0 = 3.327 \cdots$ while the best estimate previously given seems to have been $\nu_0 \leqq 32^{1/2} = 5.657 \cdots$.

The object of this note is to determine the best constant $\mu = \mu_0$ satisfying the original van der Corput inequality (1). First it will be shown that the two problems are actually equivalent and in fact $\mu_0 = \nu_0$. Furthermore, a maximum for the expression on the left of (1), under the restriction $f''(x) \geqq r$, is taken if the same function $f(x)$ and interval $[a, b]$ are chosen as in the case

* Received February 10, 1938.

[1] Cf. E. Landau, *Vorlesungen über Zahlentheorie*, vol. 2 (1927), p. 60; or E. Landau, *Einfuhrung in die Differentialrechnung und Integralrechnung* (1934), p. 307.

[2] R. Kershner, "Determination of a van der Corput-Landau absolute constant," *American Journal of Mathematics*, vol. 57 (1935), pp. 840-846.

(2). Secondly, the determination of the best constant in (1) will be carried out directly, without reference to the equivalent problem connected with (2). In view of this equivalence, the direct treatment of the case (1) to be given here may be considered as an alternative to the author's previous treatment of (2). In this connection it may be mentioned that, while the considerations to be used are certainly no simpler than those previously used in connection with (2) they do have the advantage of being considerably more intuitive geometrically. Furthermore, the method to be used in the direct treatment of (1) seems to be of interest in itself and is expected to have further applications in the study of exponential integrals.

THEOREM I. *Let μ_0, ν_0 denote the least values of μ, ν satisfying* (1), (2) *respectively, for all functions $f(x)$ with $f''(x) \geqq r > 0$. Then $\mu_0 = \nu_0$.*

The trivial inequality

$$\left| \int_a^b \exp(if(x))\,dx \right| \geqq \left| \int_a^b \cos f(x)\,dx \right| \tag{4}$$

shows that $\mu_0 \geqq \nu_0$ so it is only necessary to show that $\mu_0 \leqq \nu_0$. Let $f(x)$ satisfy the conditions of the van der Corput lemma. Then

$$\int_a^b \exp(if(x))\,dx = \left| \int_a^b \exp(if(x))\,dx \right| \exp(i\omega)$$

where $\omega = \arg\left[\int_a^b \exp(if(x))\,dx\right]$ is a constant. Thus

$$\int_a^b \exp i(f(x) - \omega)\,dx = \left| \int_a^b \exp(if(x))\,dx \right|. \tag{5}$$

This shows that the left-hand side of (5) is real, and in fact positive, so that

$$\int_a^b \exp[i(f(x) - \omega)]\,dx = \left| \int_a^b \cos(f(x) - \omega)\,dx \right|. \tag{6}$$

From (5) and (6)

$$\left| \int_a^b \exp(if(x))\,dx \right| = \left| \int_a^b \cos(f(x) - \omega)\,dx \right| \tag{7}$$

so that $\mu_0 \leqq \nu_0$.

Comparison of the equality (7) with the above mentioned result concerning (2) makes the following statement obvious:

THEOREM II. *The expression on the left of* (1) *attains a maximum,*

under the restriction $f''(x) \geqq r > 0$, if the function $f(x)$ is chosen to be the parabola $f(x) = rx^2/2$, and if $-a = b = [(\pi - 2c_0)/r]^{1/2}$.

Theorem II will now be demonstrated by a direct geometrical argument. Consider the curve $S(f)$, represented in the complex plane by

$$z = x + iy = \int_0^s \exp(if(x))\,dx$$

or with parametric equations

$$(8) \qquad \begin{aligned} x &= \int_0^s \cos f(x)\,dx \\ y &= \int_0^s \sin f(x)\,dx. \end{aligned}$$

Since $(dx/ds)^2 + (dy/ds)^2 = 1$, the parameter s is actually the arc-length along the curve $S(f)$. Also

$$dy/dx = \tan f(s)$$

so that $f(s)$ represents the inclination of $S(f)$ at the point s (i. e., at the point with arc-length s) and $f'(s)$ is the curvature of $S(f)$ at the same point.

It will be assumed that $f'(s)$ takes on all values between $-\infty$ and $+\infty$, since otherwise the region of definition of the function $f(x)$ can be extended. Also it will be assumed that $f(0) = f'(0) = 0$, a normalization that involves no loss of generality. Thus the curve $S(f)$ passes through the origin when $s = 0$ and is tangent to the x-axis at this point. Furthermore the curvature $k(s) = f'(s)$ of $S(f)$ vanishes at this point. For $s < 0$ the curvature $k(s)$ is negative and for $s > 0$, $k(s) > 0$. Finally $k'(s) = f''(s) \geqq r > 0$ by assumption.

It is easy to see that the curve $S(f)$ is a double spiral without double points (see figure p. 552). In fact, if the osculating circle be drawn at the point $s = a > 0$ then all points of the curve $S(f)$ for which $s > a$ lie within this circle while all points $s < a$ lie outside this circle. If $a < 0$ the situation is reversed. Finally, at $a = 0$ there is no osculating circle but the two arcs of $S(f)$ corresponding to $s > 0$ and $s < 0$ lie on opposite sides of the tangent line, i. e., of the x-axis.

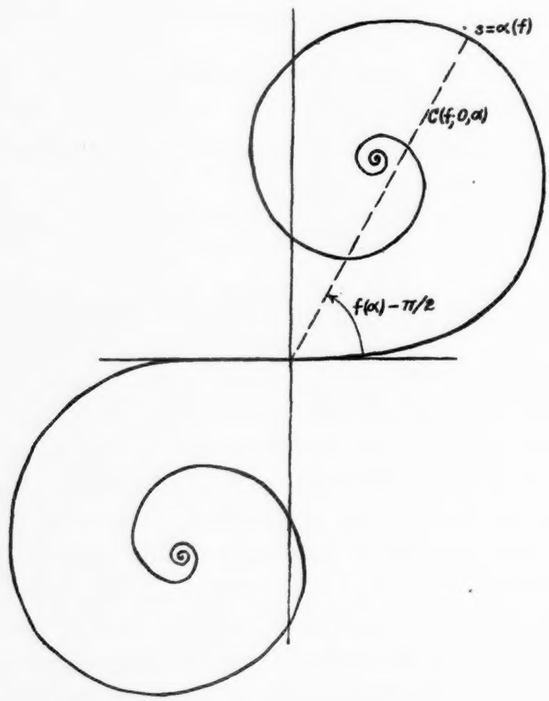
Now $\left| \int_a^b \exp(if(x))\,dx \right|$ is simply the length of the chord joining the points $s = a$, $s = b$ on the curve $S(f)$. The notation $C(f; a, b)$ will be used to denote this chord or its length. Thus the problem is to determine the maximum of $C(f; a, b)$ for all spirals $S(f)$.

First it will be shown that it is sufficient to consider chords which pass through the origin and spirals which are symmetrical with respect to the origin. Suppose that a and b are named in such a way that $C(f; 0, a) \geqq C(f; 0, b)$, $a \neq 0$. Then, from the triangle inequality

$$C(f; a, b) \leqq C(f; 0, a) + C(f; 0, b)$$

it is seen that

$$2C(f; 0, a) \geqq C(f; a, b). \tag{9}$$



Let $g(x)$ be defined by $g(x) = f(x)$ if $\operatorname{sign}(x) = \operatorname{sign}(a)$, $g(-x) = g(x)$ and $g(0) = 0$. Then $S(g)$ is symmetrical with respect to the origin and there exists on $S(g)$ a chord $C(g; -a, a)$ such that

$$C(g; -a, a) = 2C(f; 0, a) \geqq C(f; a, b).$$

Thus, to find a maximum for all chords $C(f; a, b)$ it is sufficient to find a maximum for $2C(f; 0, a)$ where $a > 0$.

Now for a given $S(f)$ there does exist a maximum $C(f; 0, a)$ and this

maximum chord is, of course, perpendicular to the curve $S(f)$ at the point a. In fact, it is easily seen from the shape of $S(f)$ that the maximum $C(f;0,a)$ occurs when a is the smallest value for which $C(f;0,a)$ is perpendicular to the curve $S(f)$ at the point a. Let this value a be denoted by $a = \alpha = \alpha(f)$. The value $a = \alpha$ clearly occurs when the inclination $f(\alpha)$ is between $\pi/2$ and $3\pi/2$. Finally, the closed curve consisting of the arc of $S(f)$ for which $0 \leqq s \leqq \alpha$ and the chord $C(f;0,\alpha)$ is a convex curve which will be denoted by $\Gamma(f;\alpha)$.

Now it will be shown that the maximum chord $C(f;0,\alpha)$ occurs when $f(x) = f_0(x) = rx^2/2$. It should be mentioned that $S(f_0)$ is the well known spiral of Cornu. Let β be such that $f_0(\beta) = f(\alpha)$. In conformity with the above let $\Gamma(f_0,\beta)$ denote the closed curve (also obviously convex) consisting of the arc of $S(f_0)$ for which $0 \leqq s \leqq \beta$ and the chord $C(f_0;0,\beta)$.

Now $f''(x) \geqq r > 0$ and $f''_0(x) \equiv r$. Also $f(0) = f_0(0) = 0$; $f'(0) = f'_0(0) = 0$. It has been shown,[3] under these conditions, that $f'(x_1) \geqq f'_0(x)$ if x_1 is the uniquely determined value $x_1 = x_1(x)$ for which $f(x_1) = f_0(x)$. Interpreted geometrically this means that the curvature $k_0(s)$ of $S(f_0)$ at any point s is not greater than the curvature $k(s_1)$ of $S(f)$ at the point s_1 where the inclination $f(s_1)$ is the same as the inclination $f_0(s)$ of $S(f_0)$ at s.

Let ϕ denote the normal inclination of $\Gamma(f;\alpha)$ or $\Gamma(f_0;\beta)$ and let $K(\phi)$, $K_0(\phi)$ represent the curvature of these curves where the curvature exists, i. e., up to the ϕ-intervals corresponding to the corners, so that

$$K(\phi) = k(s(\phi)),\ \ K_0(\phi) = k_0(s_0(\phi))\,;\ \ -\pi/2 < \phi < f_0(\beta) - \pi/2.$$

Then, according to the preceding paragraph,

$$K(\phi) \geqq K_0(\phi)\,;\quad -\pi/2 < \phi < f_0(\beta) - \pi/2. \tag{10}$$

A well known theorem[4] on convex curves states that if the inequality occurring in (10) holds for all ϕ-values then the curve $\Gamma(f;\alpha)$ can be placed within $\Gamma(f_0;\beta)$. Slight modifications of the proof of this theorem, as given *loc. cit.*[4], show that if $H(\phi)$, $H_0(\phi)$ are the supporting functions[4] of $\Gamma(f;\alpha)$, $\Gamma(f_0;\beta)$ respectively then

$$H(f(\alpha) - \pi/2) = H(f_0(\beta) - \pi/2) \leqq H_0(f_0(\beta) - \pi/2). \tag{11}$$

But, by definition of the supporting function,

$$H(f(\alpha) - \pi/2) = C(f;0,\alpha) \quad \text{and} \quad H_0(f_0(\beta) - \pi/2) = C(f_0;0,\beta).$$

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[3] *Loc. cit.*[2], p. 845.

[4] Cf. W. Blaschke, *Kreis und Kugel* (1916), pp. 115-116.

Thus, from (11),

$$C(f; 0, \alpha) \leqq C(f_0; 0, \beta). \tag{12}$$

It is now seen from (9) and (12) that

$$\begin{aligned} C(f; a, b) &\leqq 2 \max C(f; 0, a) = 2C(f; 0, \alpha) \\ &\leqq 2 \max C(f_0; 0, c) = \max C(f_0; -c, c). \end{aligned}$$

Expressed in the integral form, with the definition of $f_0(x)$ substituted, this last inequality becomes

$$\left| \int_a^b \exp(if(x))\,dx \right| \leqq \max \left| \int_{-c}^{c} \exp(irx^2/2)\,dx \right|.$$

The proof of Theorem II is completed by the trivial determination of c.

THE UNIVERSITY OF WISCONSIN.

A PROPERTY OF SPHERICAL HARMONICS.*

By Hans Lewy.

In this paper will be proved the following theorems:

Theorem 1. *Let $u(x,y,z)$ be a homogeneous polynomial of degree $n > 2$ whose range of values covers positive and negative numbers and whose Hessian $H_{[u]}$ vanishes only at $x = y = z = 0$. Then those points of the unit sphere σ $(x^2 + y^2 + z^2 = 1)$ for which $u = 0$, form two distinct closed convex curves*[1] *without singularities and diametral to each other.*

Theorem 2. *Let $u(x,y,z)$ be a spherical harmonic of degree $n > 2$, i.e. a regular solution of*

$$u_{xx} + u_{yy} + u_{zz} = 0$$

and of

$$xu_x + yu_y + zu_z = nu$$

which does not vanish identically. Then there exists a point distinct from the origin where the Hessian

$$H_{[u]} \equiv \begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{xy} & u_{yy} & u_{yz} \\ u_{xz} & u_{yz} & u_{zz} \end{vmatrix}$$

vanishes.

We employ the following

Lemma 1. *If $v(x,y,z)$ be a homogeneous polynomial of degree $m > 1$ and (x_0, y_0, z_0) an arbitrary point where $v(x_0, y_0, z_0) \neq 0$, then the Gaussian curvature of the surface $v(x,y,z) = v(x_0, y_0, z_0)$ has the same sign as $vH_{[v]}$ and vanishes where $H_{[v]}$ vanishes.*

Proof of the Lemma. $|v|^{1/m}$ is of degree 1 and analytic wherever $v \neq 0$. The first derivatives of $|v|^{1/m}$ are of degree 0. On writing down the homogeneity relations for these, we obtain three homogeneous equations whose determinant vanishes. This determinant is $H_{[|v|^{1/m}]}$ and we have for $v \gtrless 0$ resp.

* Received March 14, 1938.

[1] A closed curve without double points is called convex if either its "interior" or its "exterior" is convex, i. e. contains, together with two points P and Q, the smaller arc of the great circle through P and Q.

$$0 = H_{[|v|^{1/m}]}$$

$$= \pm \begin{vmatrix} v_{xx} + (1/m - 1)\dfrac{v_x^2}{v}, \cdots & , \cdots \\ v_{xy} + (1/m - 1)\dfrac{v}{v_x v_y}, \; v_{yy} + (1/m - 1)\dfrac{v_y^2}{v}, \cdots \\ \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot, \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot, \cdots \end{vmatrix} \cdot \frac{|\, v\,|^{3(1/m-1)}}{m^3}$$

$$= \pm \frac{|\, v\,|^{3(1/m-1)}}{m^3} \{H_{[v]} - (1/v)(1 - 1/m)[(v_{xx}v_{yy} - v_{xy}^2)v_z^2 + \cdots]\}.$$

The expression $(v_{xx}v_{yy} - v_{xy}^2)v_z^2 + \cdots$ differs from the Gaussian curvature of $v = \text{const.}$ by a positive factor, and our Lemma follows.

Proof of Theorem I. Replacing, if necessary, u by $-u$, we may assume that $H_{[u]} > 0$ and that the curvature of the surfaces S_+ ($u = \text{const.} > 0$) is positive, the curvature of the surfaces S_- ($u = \text{const.} < 0$) is negative. Furthermore it is legitimate to suppose n even, since for n odd the Hessian is an odd function and the conditions of our Theorem cannot be satisfied. Since $H_{[u]} \neq 0$, the curves $u = 0$ on σ are without singularities and possess a non-vanishing geodetic curvature at each point (i. e. have no contact of higher order with a great circle). For it is a familiar fact of algebraic geometry that $H_{[u]} \neq 0$ implies that the projection of $u = 0$ from the origin O on the tangent plane T of σ at a point P of $u = 0$ has no singularity at P and its curvature at P does not vanish; for the coördinates (x, y, z) on σ can be interpreted as projective coördinates of the projection on T.

Consider that branch of $u = 0$ which passes through P. It must be a closed spherical curve since it does not possess any singularities, and its geodetic curvature does not vanish. Its "interior," therefore, is a convex domain B distinct from a hemisphere.[2]

We show $u > 0$ in B. Assume that a suitable rotation of the system of coördinates transforms the above point P into $(1, 0, 0)$ and the direction of $u = 0$ at P into $dx = dz = 0$. We conclude from the homogeneity of u that at P for $dx = dz = 0$, $dy \neq 0$

$$0 = u = u_x, \quad xu_{xx} = u_{xx} = 0, \quad u_x dx + u_y dy + u_z dz = du = u_y dy = 0,$$
$$u_y = 0, \quad xu_{xy} = u_{xy} = 0.$$

[2] I owe to M. W. Fenchel the following information regarding this theorem about spherical curves of positive geodetic curvature: the theorem, though not explicitly stated in literature, follows immediately from i) a theorem by Möbius that a positively curved arc without singularities nor double points is contained in a hemisphere (Möbius, *Werke II*, pp. 183-187), and ii) the corresponding fact about planar domains bounded by convex curves to which it may be reduced by central projection.

By hypothesis

$$H_{[u]} = \begin{vmatrix} 0 & 0 & u_{xz} \\ 0 & u_{yy} & u_{yz} \\ u_{xz} & u_{yz} & u_{zz} \end{vmatrix} = - u_{xz}{}^2 u_{yy} > 0$$

whence

$$u_{yy} < 0, \quad u_{xz} \neq 0.$$

Now we have along $u = 0$ at P

$$dz = 0, \quad 0 = d^2u/dy^2 = u_{yy} + u_z d^2z/dy^2$$

where y is taken as independent variable. Thus $u_z d^2z/dy^2 > 0$, and u is positive on that side of a neighborhood of $u = 0$ against which $u = 0$ is concave and negative on the other side.

If $u = 0$ at P then $u = 0$ at the diametral point P'. But P and P' cannot lie on the same branch of $u = 0$. For choose in the neighborhood of P $(1, 0, 0)$ two points P_1 and P_2 on $u = 0$ with small $|y_{P_1}|$, $|y_{P_2}|$ and $y_{P_1} < 0$, $y_{P_2} > 0$. B is contained within a loon formed by the two great circles tangent to $u = 0$ at P_1 and P_2. The condition: $d^2z/dy^2 \neq 0$ on $u = 0$ in the neighborhood of P excludes that either circle passes through P. On the other hand P lies within the loon, and consequently P' lies outside of it and of B.[3]

Thus the number of disconnected branches of $u = 0$ is even and each branch bounds a convex domain in which $u > 0$. For suppose $u = 0$ inside of a branch C of $u = 0$, on a certain set of points E. Choose in E a point Q of minimum distance from C. Since $u > 0$ in the neighborhood of C for all points inside of C, Q has a positive distance from C. Thus there would exist a convex curve C', passing through Q and contained within C, in whose exterior necessarily $u < 0$ for a neighborhood of C'. This leads to a contradiction with the assumption that Q is a point of E of minimum distance from C.

Hence the domain $u < 0$ on σ consists of σ minus an even number of disjoint convex domains.

Consider again an arbitrary point of $u = 0$ in the above notation: $P(1, 0, 0)$, $u = u_x = u_{xx} = u_{xy} = u_y = 0$, $u_{xz} \neq 0$. Denote by $Q'(1, 0, z)$ a point for which $|z|$ is small and $u < 0$. The asymptotic directions of the surface $u =$ const. that passes through Q' are determined by

$$(1) \qquad u_x dx + u_y dy + u_z dz = 0$$

$$(2) \qquad u_{xx} dx^2 + 2u_{xy} dx dy + u_{yy} dy^2 + 2u_{xz} dx dz + \cdots = 0.$$

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[3] Also cf. Footnote 2.

The second relation expresses that $d^2x = d^2y = d^2z = 0$ imply $d^2u = 0$. On replacing dx by its value in (1) and substituting in (2) we obtain

$$\begin{aligned} 0 = A \equiv\ & u_x(u_{yy}dy^2 + 2u_{yz}dydz + u_{zz}dz^2) \\ & + (u_ydy + u_zdz)\left[(u_{xx}/u_x)(u_ydy + u_zdz) - 2u_{xy}dy - 2u_{xz}dz\right]. \end{aligned}$$

As Q' tends to P $(z \to 0)$, this equation reduces to $(dz/dy)^2 = 0$. For

$$\begin{aligned} \lim_{z\to 0} \frac{u_{xx}u_z - 2u_xu_{xz}}{u_x} &= \frac{u_{xxz}u_z + u_{xx}u_{zz} - 2u_{xzz}u_x - 2u_{xz}u_{xz}}{u_{xz}}\Bigg|_{z=0,\,x=1,\,y=0} \\ &= \frac{u_z^2[(n-1)(n-2) - 2(n-1)^2]}{(n-1)u_z}\Bigg|_{x=1,\,y=0,\,z=0} \\ &= -nu_z(1,0,0), \end{aligned}$$

$$\lim_{z\to 0} \frac{u_{xx}u_y - u_{xy}u_x}{u_x} = 0.$$

Hence the projections of the asymptotic directions at Q' on T tend to the direction of the tangent of $u = 0$ at P as $Q' \to P$. The same result would have been obtained if, instead of $Q'(1, 0, z)$, we had considered the point $\bar{Q}$ $(1/\sqrt{1+z^2}, 0, z/\sqrt{1+z^2})$ which is the intersection of OQ' with σ. For the asymptotic directions are determined by the equations (1) and (2) which are homogeneous in x, y, z.

We next consider the projections from O on the sphere σ of the asymptotic directions at $\bar{Q} = \bar{Q}(\bar{x}, 0, \bar{z})$, given by

$$(d'x, d'y, d'z) = (dx, dy, dz) - (\bar{x}dx + \bar{z}dz)(\bar{x}, 0, \bar{z})$$

with (dx, dy, dz) satisfying (1) and (2). We have

$$\lim_{\bar{z}\to 0} u_x(\bar{x}, 0, \bar{z})/\bar{z} = u_{xz}(1, 0, 0),$$

whence

$$\bar{z}\frac{dx}{dy} = -\bar{z}\,\frac{u_ydy + u_zdz}{u_xdy} \to \lim_{\bar{z}\to 0} \frac{dz}{dy} \cdot \frac{1}{n-1}.$$

Since $\lim\limits_{\bar{z}\to 0} \dfrac{dz}{dy} = 0$, we conclude

$$\frac{d'z}{d'y} = \frac{dz - (\bar{x}dx + \bar{z}dz)\bar{z}}{dy} \to 0$$

and

$$\frac{d'x}{d'y} = \frac{dx - (\bar{x}dx + \bar{z}dz)\bar{x}}{dy} \to \lim_{\bar{z}\to 0} \frac{(1-\bar{x}^2)dx}{dy} = \lim_{\bar{z}\to 0} \bar{z}^2\frac{dx}{dy} = 0.$$

Thus is shown that the projections on σ of the asymptotic directions at points $\bar{Q}$ of σ with $u_{\bar{Q}} < 0$ form a field of two vectors which tend to the tangent of C as $\bar{Q}$ approaches C on a normal. It is easily seen that the above limit considerations hold uniformly for all points of C as the higher derivatives of u

on σ are uniformly bounded. The fields of the two vectors is therefore defined and continuous in the domains $u \leqq 0$.

Choose an arbitrary point N of σ for which $u_N > 0$ and project the vector field stereographically from N on a plane. Of the images of the spherical curves $u = 0$, one will contain the image of the domains $u < 0$ while the images of the other curves $u = 0$ will constitute interior boundaries of the domain of definition of the projected field vectors. As the field is without singularities the sum of the rotations of the vectors over the interior boundaries equals that over the exterior boundary. Evidently the contribution of each boundary is 4π. Hence there can be only one interior boundary. In other words, the curves $u = 0$ on σ are precisely two diametrally opposite convex curves without singularities.

The remaining paragraphs will show that for no spherical harmonic u of degree $n > 2$, n even, the curves $u = 0$ on σ can be two distinct diametrally opposite convex curves without singularities.

LEMMA 2. *Consider on σ a closed convex curve C_1 different from a great circle. Denote by B_1 the interior of C_1, and by B_2 the interior of the curve C_2 diametrally opposite to C_1. Then it is possible to locate on σ a circular domain of spherical radius 33° in such a way that its closure lies either in B_1 or in $\sigma - (B_1 + C_1) - (B_2 + C_2)$.*

We omit a proof of this Lemma and refer instead to a note by R. M. Robinson [4] in which is shown that one always can locate in the described way a circular domain of radius $<$ arc tan $\frac{1}{2}\sqrt{3}$ and that this is the best possible upper bound.

LEMMA 3. *Let u be a spherical harmonic of degree $n \geqq 4$. Then each circular domain β of the unit sphere of radius 33° contains a zero of u.*

Suppose there exists a spherical harmonic u of degree $n \geqq 4$ which is positive in a circle β of radius 33° on σ. Introduce polar coördinates (r, θ, ϕ) so that the center of β corresponds to $r = 1$, $\theta = 0$. Consider

$$v(r, \theta) = \int_0^{2\pi} u(r, \theta, \phi)\, d\phi$$

$v(r, \theta)$ is a spherical harmonic of degree n, of the form $cr^n P_n(\cos\theta)$, $c > 0$, studied by Legendre.

We have

$$P_\nu(1) = 1,\ \nu = 0, 1, \cdots; \quad P_0(\xi) \equiv 1, \quad P_1(\xi) \equiv \xi,$$

[4] R. M. Robinson, *Bulletin of the American Mathematical Society*, 1938, pp. 115-116.

and the recursion formula

$$(\nu+1)P_{\nu+1}(\xi) - (2\nu+1)\xi P_\nu(\xi) + \nu P_{\nu-1}(\xi) = 0, \qquad \nu \geqq 1.$$

Denote by $\xi_\nu < 1$ the largest value of ξ for which $P_\nu(\xi) = 0$. By the recursion formula $P_{\nu+1}(\xi_\nu) < 0$ and

$$0 = \xi_1 < \xi_2 < \xi_3 \cdots.$$

Furthermore

$$8P_4(\xi) = 35\xi^4 - 30\xi^2 + 3,$$

$$\xi_4{}^2 = \frac{15+\sqrt{120}}{35} > \frac{15+10.6}{36} \geqq \frac{16^2 \cdot 10}{6^2 \cdot 100} \geqq 0.843^2.$$

Thus we have for $0 < \theta_4 < 90°$ and $\xi_4 = \cos\theta_4$ the inequality

$$\cos\theta_4 > 0.843, \qquad \theta_4 < 33°.$$

Consequently, for $\xi_\nu = \cos\theta_\nu$, $0 < \theta_\nu < 90°$,

$$33° > \theta_4 > \theta_5 > \cdots.$$

But this makes impossible our assumption that u and hence $v(r,\theta)$ are positive in β.

The comparison of Theorem 1 with Lemmas 2 and 3 easily yields Theorem 2.

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ON THE CHARACTER OF CERTAIN ENTIRE FUNCTIONS IN DISTANT PORTIONS OF THE PLANE.*

By C. V. Newsom.

Recently Ford has proved a theorem pertaining to the behavior of the series,

$$\sum_{n=0}^{\infty} g(n) z^n; \text{ radius of convergence } = \infty,$$

when the absolute value of z becomes large. In his study the coefficient $g(n)$, when considered as a function $g(w)$ of the complex variable $w = x + iy$, must satisfy the two following conditions throughout any arbitrary right-half plane $x > x_0$:

(a) is single valued and analytic,
(b) is such that for all $|y|$ sufficiently large one may write

$$|g(x + iy)| < Ke^{(\pi+\epsilon)|y|},$$

where ϵ is an arbitrarily small positive quantity given in advance and where K depends only upon x_0 and ϵ.[1]

The present paper studies a similar problem in which, however, condition (b) given above is somewhat less restricted. The theorem to be established is as follows:

Theorem. *Let it be assumed that the coefficient $g(n)$ occurring in the general term of the power series,*

$$\sum_{n=0}^{\infty} g(n) z^n; \textit{ radius of convergence } = \infty, \tag{1}$$

may be regarded as a function $g(w)$ of the complex variable $w = x + iy$ and as such satisfies the following two conditions:

(a) *is single valued and analytic throughout the finite w-plane,*
(b) *is such that for all values of x and y one may write*

* Received December 14, 1937.

[1] Ford, *The Asymptotic Developments of Functions Defined by Maclaurin Series*, (Scientific Series, University of Michigan, vol. 11), 1936, pp. 30-37. For a further discussion of developments in this field, see the same treatise.

$$(2) \qquad | g(x+iy) | < Ke^{k\pi|y|},$$

where K is a constant independent of x and y, and k is any given positive integer.

Then the function $f(z)$ defined by series (1) when considered for all values of z satisfying the condition, $-\pi < \arg \pm z < \pi$, may be expressed in the form

$$(3) \quad f(z) = \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x)[\pm z]^x \frac{\sin k\pi x}{\sin \pi x} \right\} dx - \sum_{m=-l}^{-1} g(m) z^m + \xi_k(l, z);$$

wherein l is any arbitrary positive integer and the upper or lower of the signs $\pm$ is to be taken according as k is odd or even. Moreover, the expression $\xi_k(l, z)$ is such that

$$(4) \qquad \lim_{|z| \to \infty} z^l \xi_k(l, z) = 0,$$

irrespective of the value chosen for l.

Proof of Theorem. The proof of this theorem is based upon the following statement which is seen to be a consequence of Cauchy's integral theorem in the calculus of residues; namely, if $P(w)$ and $Q(w)$ are any two functions of the complex variable $w = x + iy$ both of which are single valued and analytic throughout a region A of the w-plane and of which $Q(w)$ vanishes within A only at the points $w = \lambda_1, \lambda_2, \cdots, \lambda_n$ which are zeros of the first order, and if c_n denotes any closed contour lying within A and including the points $w = \lambda_1, \lambda_2, \cdots, \lambda_n$, then one may write

$$(5) \quad \frac{1}{2\pi i} \int_{c_n} \frac{P(w)}{[Q(w)]^{k+1}} dw = \sum_{n=1}^{n} \frac{1}{k!} \left\{ \frac{d^k}{dw^k} \left[P(w) \left(\frac{w - \lambda_n}{Q(w)} \right)^{k+1} \right] \right\}_{w=\lambda_n},$$

in which k denotes any positive integer. The general term of the summation upon the right is the residue at $w = \lambda_n$ of the integrand upon the left as may be readily verified by observation of its Laurent's expansion.

If the formula of Leibnitz be employed upon the derivative of the product within the bracket of (5), it follows that

$$(6) \quad \frac{1}{2\pi i} \int_{c_n} \frac{P(w)}{[Q(w)]^{k+1}} dw$$
$$= \sum_{n=1}^{n} \frac{1}{k!} \sum_{r=0}^{k} \frac{k!}{r!(k-r)!} \left[\frac{d^{k-r}}{dw^{k-r}} P(w) \right]_{w=\lambda_n} \left[\frac{d^r}{dw^r} \left(\frac{w-\lambda_n}{Q(w)} \right)^{k+1} \right]_{w=\lambda_n}.$$

For the proof of the theorem of this paper, we shall choose $Q(w) = \sin \pi w$.

With this choice, $\lambda_n = n$ where n is any positive integer although the analysis follows also when the region A is extended to include a set of negative integers.

Through the medium of elementary transformations, it is found that

$$\left[\frac{d^r}{dw^r}\left(\frac{w-n}{\sin \pi w}\right)^{k+1}\right]_{w=n} = \frac{(-1)^{kn+n}\pi^r}{\pi^{k+1}}\left[\frac{d^r}{dw^r}\left(\frac{w}{\sin w}\right)^{k+1}\right]_{w=0}.$$

Hence it follows at once that

$$\left[\frac{d^r}{dw^r}\left(\frac{w-n}{\sin \pi w}\right)^{k+1}\right]_{w=n} = \frac{(-1)^{kn+n}\pi^r(k-r)!\,r!}{\pi^{k+1}\cdot k!}\Sigma\alpha_1\alpha_2\cdots\alpha_r, \tag{7}$$

where $\Sigma\alpha_1\alpha_2\cdots\alpha_r$ denotes the sum of the $\binom{k}{r}$ products of r factors each formed by taking the possible combinations of the k quantities, $\pm(k-1)i$; $\pm(k-3)i;\cdots,\left\{\begin{matrix}\pm i\\0\end{matrix}\right\}$, r at a time; i having the usual interpretation as the imaginary unit, and where $\left\{\begin{matrix}\pm i\\0\end{matrix}\right\}$ is understood as $\pm i$ or 0 according as k is even or odd.[2]

Thus the general term of the summation in (6) with respect to n becomes:

$$\frac{(-1)^{kn+n}}{k!\,\pi^{k+1}}\sum_{r=0}^{k}[(\Sigma\,\alpha_1\alpha_2\cdots\alpha_r)\pi^r P^{(k-r)}(n)],$$

where $P^{(k-r)}(n)$ indicates the $(k-r)$ derivative of $P(w)$ evaluated at $w=n$.

It is now desired that this expression should equal $\dfrac{g(n)z^n}{k!\,\pi^{k+1}}$ where $g(n)z^n$ is the n-th term of the given series (1). In order that this may be the case it is evidently necessary and sufficient that the hitherto undetermined function $P(n)$ shall satisfy the following differential equation in n:

$$\sum_{r=0}^{k}[(\Sigma\,\alpha_1\alpha_2\cdots\alpha_r)\pi^r P^{(k-r)}(n)] = g(n)[\pm z]^n, \tag{8}$$

wherein the upper or lower of the signs $\pm$ is to be taken according as k is odd or even.

This non-homogeneous differential equation is well adapted to elementary methods of solution. It has constant coefficients and the roots of its characteristic equation are distinct. Moreover, the roots are observed to be the quantities

$$\pm(k-1)\pi i,\ \pm(k-3)\pi i,\cdots,\left\{\begin{matrix}\pm\pi i\\0\end{matrix}\right\}. \tag{9}$$

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[2] For a proof of this fact see Newsom, *The American Mathematical Monthly*, vol. 38, no. 9, pp. 500-504.

Henceforth these roots will be denoted by $r_1\pi i$, $r_2\pi i$, etc. where $r_j = k + 1 - 2j$; $1 \leq j \leq k$. Then by a well known formula, a particular solution of (8) may be written directly as follows:

$$P(n) = \sum_{j=1}^{k} b_j e^{r_j\pi i n} \int_0^n e^{-r_j\pi i n} g(n)[\pm z]^n dn, \tag{10}$$

where

$$b_j = \frac{1}{(\pi i)^{k-1}} \prod_{a=1}^{j-1} \frac{1}{(r_j - r_a)} \prod_{a=j+1}^{k} \frac{1}{(r_j - r_a)}; \quad 1 \leq j \leq k; \tag{11}$$

in which

$$\prod_{a=p}^{q} \frac{1}{(r_j - r_a)} = 1 \text{ if } p \geqq q.$$

Having determined $P(n)$ in this manner, we observe that if n be now replaced by the complex variable $w = x + iy$ the result is an analytic function $P(w)$ of w throughout the finite w-plane.

If then in (6) the path of integration c_n be chosen as any closed (finite) contour which encloses the points $w = -l, -l+1, -l+2, \cdots, -1, 0, 1, 2, \cdots, n$; l being any given positive integer; and if we employ the indicated form of the function $P(w)$, equation (6) becomes

$$\frac{1}{2\pi i} \int_{c_n} \frac{P(w)}{[Q(w)]^{k+1}} dw = \frac{1}{k!\,\pi^{k+1}} \sum_{n=-l}^{n} g(n) z^n. \tag{12}$$

With the actual forms for $P(w)$ and $Q(w)$ substituted into (12), the relation becomes

$$\sum_{n=0}^{n} g(n) z^n = \frac{k!\,\pi^k}{2i} \sum_{j=1}^{k} b_j \int_{c_n} \frac{e^{r_j\pi i w} \int_0^w e^{-r_j\pi i w} g(w)[\pm z]^w dw}{(\sin \pi w)^{k+1}} dw \tag{13}$$
$$- \sum_{m=-l}^{-1} g(m) z^m.$$

Thus we arrive at an expression in the form of a contour integral for the sum of the first n terms of the given series (1).

The study of the right-hand member of (13) involves a consideration of k integrals of the form,

$$\int_{c_n} \frac{e^{r_j\pi i w} \int_0^w e^{-r_j\pi i w} g(w)[\pm z]^w dw}{(\sin \pi w)^{k+1}} dw. \tag{14}$$

Within this contour integral we shall designate

(15) $$I_j(w, z) = \int_0^w e^{-r_j\pi i w} g(w)[\pm z]^w dw.$$

Furthermore, throughout the discussion which follows $[\pm z]$ will be regarded as in the form,

$$[\pm z] = \rho(\cos\phi + i\sin\phi).$$

Moreover, $[\pm z]^w$, when evaluated for any special values of z and $w = x + iy$, is rendered precise in meaning through the following convention:

(16) $$[\pm z]^w = \rho^x e^{-\phi y} e^{(y\log\rho + x\phi)i}; \qquad -\pi < \phi \leqq \pi.$$

Then it follows that

(17) $$I_j(w, z) = \int_0^w \rho^x e^{(r_j\pi - \phi)y} e^{[y\log\rho + (\phi - r_j\pi)x]i} g(w)\,dw.$$

For the path of integration to be employed in (17) let us choose that one which, starting at the origin $w = 0$, proceeds along the real axis to the point $w = x$ and then proceeds parallel to the pure imaginary axis to the point $w = x + iy$ in question. Then $I_j(w, z)$ takes the form,

(18) $$I_j(w, z) = R_j(\rho, \phi, x) + S_j(\rho, \phi, x, y),$$

where

(19) $$R_j(\rho, \phi, x) = \int_0^x \rho^x e^{(\phi - r_j\pi)xi} g(x)\,dx$$

and

(20) $$S_j(\rho, \phi, x, y) = i\rho^x \int_0^y e^{(r_j\pi - \phi)y} e^{[y\log\rho + (\phi - r_j\pi)x]i} g(x + iy)\,dy.$$

By virtue of condition (2) postulated upon $g(w)$, if $y > 0$, it is known that

$$|g(x + iy)| < Ke^{k\pi y}.$$

Thus it follows that

(21) $$|S_j(\rho, \phi, x, y)| < K\rho^x \int_0^y e^{[k\pi + r_j\pi - \phi]y} dy$$
$$< K\rho^x \frac{e^{[k\pi + r_j\pi - \phi]y}}{[k\pi + r_j\pi - \phi]},$$

if the equality sign given in the range of ϕ in (16) be eliminated. In similar manner, if $y < 0$, we have

(22) $$|S_j(\rho, \phi, x, y)| < K\rho^x \frac{e^{[-k\pi + r_j\pi - \phi]y}}{|[-k\pi + r_j\pi - \phi]|}.$$

Hence if, in particular, we confine z to any finite region T of the z-plane such

that ρ is bounded and within which $-\pi+\epsilon<\phi<\pi-\epsilon$, where ϵ is an arbitrarily small positive quantity, and confine x and y to any strip of finite width (thus making x bounded) drawn in the w-plane parallel to the y-axis, we see that, according as $y>0$ or $y<0$ we may write

$$(23) \qquad \begin{aligned} |S_j(\rho,\phi,x,y)| &< Me^{(k\pi+\pi+r_j\pi-\epsilon)y} \\ |S_j(\rho,\phi,x,y)| &< Me^{(-k\pi-\pi+r_j\pi+\epsilon)y}, \end{aligned}$$

where M is an assignable positive constant, independent of ρ, ϕ x, and y. Moreover, for ρ, ϕ, and x as thus bounded, observation of (19) reveals that

$$(24) \qquad |R_j(\rho,\phi,x)|<N,$$

where N is a constant independent of ρ, ϕ, and x.

The foregoing properties being premised in regard to the function $I_j(w,z)$, let us return to a further study of the integral (14). For the contour c_n let us take specifically (Compare with the remark preceding (12)) the rectangle formed in the w-plane by the lines,

$$(25) \qquad w=-l-\tfrac{1}{2}+iy,\ w=n+\tfrac{1}{2}+iy,\ w=x+ip, \text{ and } w=x+iq,$$

where l is an arbitrarily large positive integer, n is an arbitrarily large positive even integer, and p and q may be regarded respectively as any arbitrarily large positive and negative quantities.

We proceed now to consider the four contributions to the integral (14) as the indicated integration over c_n is performed in turn along the four sides of the rectangle described in (25).

First, along the side upon which $w=x+ip$ we have $dw=dx$ and $\sin\pi w=\sin\pi(x+ip)=\sinh\pi p(\sin\pi x\operatorname{ctnh}\pi p+i\cos\pi x)$. Moreover, the integration takes place from $x=n+\frac{1}{2}$ to $x=-l-\frac{1}{2}$. Hence, if the contribution to (14) from this side of c_n be designated by A_j, we have

$$(26) \qquad A_j=\int_{n+\frac{1}{2}}^{-l-\frac{1}{2}}\frac{e^{r_j\pi i(x+ip)}[R_j(\rho,\phi,x)+S_j(\rho,\phi,x,p)]}{[\sinh\pi p(\sin\pi x\operatorname{ctnh}\pi p+i\cos\pi x)]^{k+1}}\,dx,$$

where $R_j(\rho,\phi,x)$ and $S_j(\rho,\phi,x,p)$ are defined by (19) and (20). By virtue of relations (23) and (24) it is known that for all the values of ρ, ϕ, and x with which we are concerned in (26) we may write

$$|R_j(\rho,\phi,x)|<N,\quad |S_j(\rho,\phi,x,p)|<Me^{(k\pi+\pi+r_j\pi-\epsilon)p};\quad \text{where } \epsilon>0.$$

Hence, upon writing in (26) $r_j\pi i(x+ip)=r_j\pi xi-r_j\pi p$, it appears that the

absolute value of the numerator of the integrand of (26) cannot become infinite to a higher order than that of $e^{(k\pi+\pi-\epsilon)p}$ as $p \to +\infty$. The absolute value of the denominator, however, becomes infinite to as high an order as that of $e^{(k\pi+\pi)p}$ as $p \to +\infty$ if one recalls the meaning of $\sinh \pi p$ and notes that the quantity in parentheses within the denominator has an absolute value which approaches the limit unity, this being true uniformly for all values of x. Thus we conclude that

$$\lim_{p \to +\infty} A_j = 0.$$

Similarly, the contribution B_j to the integral (14) arising from the side of c_n upon which $w = x + iq$ $(q < 0)$ is found to approach the limit zero as $q \to -\infty$. In fact, this contribution is seen to be similar to (26) in form. The integration, however, in this case takes place from $x = -l - \frac{1}{2}$ to $x = n + \frac{1}{2}$ and the expression $S_j(\rho, \phi, x, q)$ which now occurs is related to the second of relations (23) instead of to the first, thus making the absolute value of the numerator of the integrand become infinite to no higher order than that of $e^{(-k\pi-\pi+\epsilon)q}$ as $q \to -\infty$, while the denominator becomes infinite like $e^{(-k\pi-\pi)q}$ as $q \to -\infty$.

Next, let us consider the contribution arising from the side $w = n + \frac{1}{2} + iy$. Here $dw = i\,dy$ and, inasmuch as n has been chosen as an even integer, we have $(\sin \pi w)^{k+1} = (\cosh \pi y)^{k+1}$. Having taken $p = +\infty$ and $q = -\infty$, the contribution to (14) from the side in question may be written

$$(27) \qquad C_j = i e^{\frac{1}{2} r_j \pi i} \left\{ R_j(\rho, \phi, n + \tfrac{1}{2}) \int_{-\infty}^{\infty} \frac{e^{-r_j \pi y}}{(\cosh \pi y)^{k+1}} dy + \int_{-\infty}^{\infty} \frac{e^{-r_j \pi y} S_j(\rho, \phi, n + \frac{1}{2}, y)}{(\cosh \pi y)^{k+1}} dy \right\}.$$

The integral,

$$\int_{-\infty}^{\infty} \frac{e^{-r_j \pi y}}{(\cosh \pi y)^{k+1}} dy,$$

which we shall denote by H_j may be evaluated directly. It evidently may be rewritten in the form

$$H_j = \frac{2^{k+1}}{\pi} \int_0^{\infty} \frac{t^{k-r_j}}{(t^2+1)^{k+1}} dt,$$

as appears by making the transformation $e^{\pi y} = t$. This new integral may be evaluated at once and is equal to

$$(28)\quad H_j = \int_{-\infty}^{\infty} \frac{e^{-r_j \pi y}}{(\cosh \pi y)^{k+1}}\, dy = \frac{(2i)^{k+1} i^{-r_j}}{(r_j - k - 1)\pi}$$
$$\prod_{\beta=1}^{j-1}\left(1 - \frac{k - r_j + 1}{2\beta}\right) \prod_{\beta=j+1}^{k}\left(1 - \frac{k - r_j + 1}{2\beta}\right), \qquad 1 \leq j \leq k;$$

in which

$$\prod_{\beta=p}^{q}\left(1 - \frac{k - r_j + 1}{2\beta}\right) = 1 \text{ if } p \geqq q.[3]$$

As to the second integral in (27) we observe from (21) and (22) and the paragraph which follows them that the absolute value of the integrand is less than

$$(29)\qquad \frac{M_1 \rho^{n+\frac{1}{2}} e^{[k\pi + \pi - \epsilon]y}}{(\cosh \pi y)^{k+1}} \quad \text{or} \quad \frac{M_1 \rho^{n+\frac{1}{2}} e^{[-k\pi - \pi + \epsilon]y}}{(\cosh \pi y)^{k+1}},$$

where M_1 is an assignable positive constant independent of ρ, ϕ, n, and y; the first or the second of these expressions to be used according as $y > 0$ or $y < 0$. In view of the fact that $(\cosh \pi y)^{k+1}$ becomes infinite like $e^{(k\pi+\pi)y}$ as $y \to +\infty$ and like $e^{(-k\pi-\pi)y}$ as $y \to -\infty$, it follows that this integral converges and converges uniformly for all values of z lying in the region T of the z-plane as described immediately under expression (22). Furthermore, we observe from (29) that if ρ be restricted to values less than unity, that is, if we consider only that portion of the region T which lies within the unit circle about the origin, then the limit of the integral in question as $n \to \infty$ will be 0. Thus, in conclusion, we may write

$$(30)\qquad C_j = i^{(1+r_j)} H_j R_j(\rho, \phi, n + \tfrac{1}{2}) + \eta_j(n, z),$$

where for the values of z under consideration, namely, for those in T such that $|z| < 1$, it follows that

$$(31)\qquad \lim_{n \to \infty} \eta_j(n, z) = 0.$$

There now remains to be considered one contribution to the integral (14), namely, the one arising from the integration over the side $w = -l - \frac{1}{2} + iy$ of the rectangle c_n. This, however, is merely what is obtained just above by using $-l - \frac{1}{2}$ instead of $n + \frac{1}{2}$ and noting that the integration now takes place from $+\infty$ to $-\infty$ instead of vice-versa. Thus this contribution takes the form,

$$(32)\qquad D_j = -i^{(1+r_j)} H_j R_j(\rho, \phi, -l - \tfrac{1}{2}) + Y_j(l, z),$$

[3] For this evaluation note formula 20, table 17, in the integral table of Bierens de Haan. It should be observed, however, that this formula introduces a removable indeterminate form when r_j is replaced by its proper value.

where

$$(33)\qquad Y_j(l,z) = -i^{(1+r_j)}\int_{-\infty}^{\infty}\frac{e^{-r_j\pi y}S_j(\rho,\phi,-l-\frac{1}{2},y)}{(\cosh \pi y)^{k+1}}\,dy.$$

As regards the integral $Y_j(l,z)$ here encountered it should be observed that an analysis similar to that which was applied to the second integral of (27) may likewise be employed here. In fact, the integrand of this integral is less in absolute value than

$$\frac{M_2\rho^{-l-\frac{1}{2}}e^{[k\pi+\pi-\epsilon]y}}{(\cosh \pi y)^{k+1}} \quad\text{or}\quad \frac{M_2\rho^{-l-\frac{1}{2}}e^{[-k\pi-\pi+\epsilon]y}}{(\cosh \pi y)^{k+1}},$$

according as $y > 0$ or $y < 0$. Thus it follows that $Y_j(l,z)$ converges uniformly throughout the portion of T in which $|z| = \rho \geqq \rho_1 > 0$, where ρ_1 is arbitrarily small; and, furthermore, is such that for those values of z in the portion of T just specified we shall have

$$(34)\qquad \lim_{|z|\to\infty} z^l Y_j(l,z) = 0.$$

We now recall from (19) that

$$R_j(\rho,\phi,x) = \int_0^x \rho^x e^{(\phi-r_j\pi)xi}g(x)\,dx,$$

in which $\rho e^{i\phi} = \pm z$, the sign $+$ or $-$ being used according as k is odd or even. Hence we may write

$$R_j(\rho,\phi,n+\tfrac{1}{2}) - R_j(\rho,\phi,-l-\tfrac{1}{2}) = \int_{-l-\frac{1}{2}}^{n+\frac{1}{2}} g(x)[\pm z]^x e^{-r_j\pi xi}dx.$$

If we now add the various contributions A_j, B_j, C_j, and D_j which make up the integral (14), we conclude that

$$(35)\qquad \int_{C_n}\frac{e^{r_j\pi iw}\int_0^w e^{-r_j\pi iw}g(w)[\pm z]^w dw}{(\sin \pi w)^{k+1}}\,dw$$
$$= i^{(1+r_j)}H_j\int_{-l-\frac{1}{2}}^{n+\frac{1}{2}} g(x)[\pm z]^x e^{-r_j\pi xi}dx + \eta_j(n,z) + Y_j(l,z),$$

wherein $\eta_j(n,z)$ and $Y_j(l,z)$ possess the properties indicated respectively in (31) and (34).

It is now possible, therefore, to restate (13) in the form,

$$(36)\qquad \sum_{n=0}^{n} g(n)z^n = \sum_{j=1}^{k} \frac{k!\,\pi^k}{2i} \cdot i^{(1+r_j)} b_j H_j$$
$$\int_{-l-\frac{1}{2}}^{n+\frac{1}{2}} g(x)[\pm z]^x e^{-r_j\pi x i} dx + \sum_{j=1}^{k} \frac{k!\,\pi^k}{2i} b_j \eta_j(n, z)$$
$$+ \sum_{j=1}^{k} \frac{k!\,\pi^k}{2i} b_j Y_j(l, z) - \sum_{m=-l}^{-1} g(m) z^m.$$

If we recall the value of b_j given in (11) and the value of H_j given in (28), it is a matter of simple algebra to determine that the entire coefficient preceding the integral within the first summation upon the right is equal to unity. Moreover it is readily shown that

$$\sum_{j=1}^{k} e^{-r_j\pi x i} = \frac{\sin k\pi x}{\sin \pi x}.$$

Hence it follows at once that

$$(37)\qquad \sum_{n=0}^{n} g(n)z^n = \int_{-l-\frac{1}{2}}^{n+\frac{1}{2}} \left\{ g(x)[\pm z]^x \frac{\sin k\pi x}{\sin \pi x} \right\} dx$$
$$- \sum_{m=-l}^{-1} g(m)z^m + \eta(n, z) + \xi_k(l, z),$$

where $\eta(n, z)$ and $\xi_k(l, z)$ possess the properties previously held respectively by $\eta_j(n, z)$ and $Y_j(l, z)$; that is, for all z in T having $|z| < 1$ we may write

$$\lim_{n\to\infty} \eta(n, z) = 0,$$

and for all z in T for which $|z| \geqq \rho_1 > 0$, $\xi_k(l, z)$ converges uniformly and

$$\lim_{|z|\to\infty} z^l \xi_k(l, z) = 0.$$

If we now confine z to values in T for which $0 < \rho_1 \leqq |z| < 1$ and then allow $n \to \infty$ in the relation (37), it follows from what has been said of $\eta(n, z)$ that we may write for all such z:

$$(38)\qquad f(z) = \sum_{n=0}^{\infty} g(n)z^n = \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x)[\pm z]^x \frac{\sin k\pi x}{\sin \pi x} \right\} dx$$
$$- \sum_{m=-l}^{-1} g(m)z^m + \xi_k(l, z).$$

In this equation, however, the left member is an analytic function of z for all finite z, as follows from the original hypothesis that the radius of convergence of the series is infinite. Likewise, the right member is an analytic

function of z for all z in T for which $|z| \geqq \rho_1 > 0$ as follows especially from the observation previously made that $\xi_k(l, z)$ converges uniformly for all such z. It follows, therefore, as the result of well known principles of analytic continuation that (38) holds true for all z in T for which $|z| \geqq \rho_1 > 0$, and the theorem as originally stated has been established.

Remarks and generalizations. It is desirable to note that in order to secure simplicity of statement, the condition (a) pertaining to the coefficient $g(n)$ of the theorem was made unnecessarily restrictive. In fact, if one examines the proof it appears readily that the following generalization may be made; namely, in case condition (a) is not satisfied but instead the function $g(w)$, while still remaining single valued throughout the finite w-plane, has m ($m \geqq 1$) singularities situated at the points $w = w_1, w_2, \cdots, w_m$, none of which are negative integers, then (3) continues to hold true provided one subtracts from the right member the sum of the residues of the function $P(w)/[Q(w)]^{k+1}$, at these points.

The same conclusion applies also in case one or more of the points w_p are negative integers, as $w = -q$ ($q \geqq 1$), except that the alterations taking place on the right in (3) must then include the deletion of the term $g(-q)/z^q$ occurring in the summation.

Also it should be remarked that condition (b) pertaining to the coefficient $g(n)$ of the theorem may be replaced by an alternate condition, namely, that $g(w)$ can be such that for all values of x and y one may write

$$\left| \frac{g(x + \frac{1}{2} + iy)}{g(x)} \right| < Ke^{k\pi|y|},$$

where K is a constant independent of x and y and k is any given positive integer $\geqq 1$. With this new condition (b) and with the other conditions as originally specified, conclusion (3) follows as before. Usually the original condition (b) is more readily applied but in certain cases when the coefficient $g(n)$ contains exponentials the alternate condition becomes desirable.

In establishing these latter remarks it should be observed that the indicated alteration upon condition (b) changes the proof of the theorem only slightly. Virtually the only essential change takes place in the expressions of (29) where the absolute value of the integrand in question now is less than

$$\frac{M_1 \,|\, g(n) \,|\, \rho^n e^{[k\pi + \pi - \epsilon]y}}{(\cosh \pi y)^{k+1}} \quad \text{or} \quad \frac{M_1 \,|\, g(n) \,|\, \rho^n e^{[-k\pi - \pi + \epsilon]y}}{(\cosh \pi y)^{k+1}},$$

where M_1 is an assignable positive constant independent of ρ, ϕ, n, and y; the first or the second of these expressions to be used according as $y > 0$ or

$y < 0$. As before, the second integral in (27) converges uniformly for the values of z lying in such a region of the z-plane as described immediately after (22). But $| g(n) | \rho^n$ is the absolute value of the n-th term of the original convergent power series (1) and thus approaches zero as $n \to +\infty$. So, without any additional restrictions upon ρ, the integral in question approaches zero as $n \to +\infty$. The remainder of the proof of the theorem readily follows without the necessity of employing the device of analytic continuation.

Remarks concerning the applications of the theorem. Due to the presence of l in the lower limit of the integral of (3), the theorem of this paper does not furnish direct information upon the asymptotic representation of the given function, $f(z)$. However, it is in regard to such asymptotic studies that the theorem appears to have important applications. These applications will apparently follow the same general procedures as previously discussed by Ford and Van Engen.[4] Since, however, for all $x > x_0$, where x_0 is arbitrary, and for all $| y |$ sufficiently large one may write

$$\frac{1}{| \Gamma(x + iy) |} < K e^{(\pi/2+\epsilon)|y|},$$

where ϵ is an arbitrarily small positive quantity given in advance, and where K depends only upon ϵ and x_0,[5] it follows that condition (b) of the theorem will be satisfied when $g(n)$ is the reciprocal of the product of any number of gamma functions of the type, $\Gamma(n + p)$. Thus in such a case the given function, $f(z)$, may be directly represented as in relation (3).

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[4] Ford, *op. cit.*, pp. 63-85.
[5] Ford, *op. cit.*, p. 61.

ON THE REPRESENTATION OF BOUNDED ANALYTIC FUNCTIONS BY SEQUENCES OF POLYNOMIALS.[*]

By O. J. Farrell.

1. **Introduction.** Let the function $f(z)$ be analytic and bounded in a region G of the plane of the complex variable z. Let $d(f, z_0)$ denote the diameter[1] of the cluster set[2] of $f(z)$ at the point z_0 on the boundary B of G. Let $D(f, B)$ denote the maximum of $d(f, z_0)$ for all points z_0 on B. A natural question to be raised is: When can $f(z)$ be represented in G by a sequence of polynomials $p_n(z), n = 1, 2, \cdots$ such that

$$\varlimsup_{n\to\infty} \{\overline{\text{bound}}\ [\ |f(z) - p_n(z)|, z \text{ in } G]\} \leqq D(f, B)\,?$$

The present paper contributes toward the solution of this problem the following two theorems.

Theorem 1. *Let the function $f(z)$ be analytic and bounded in the interior T of a limited Jordan curve J. Then there exists a sequence of polynomials $p_1(z), p_2(z), \cdots$ converging to $f(z)$ in T, uniformly on any closed set in T, so that*

$$\varlimsup_{n\to\infty} \{\overline{\text{bound}}\ [\ |f(z) - p_n(z)|, z \text{ in } T]\} \leqq D(f, J). \tag{1}$$

Theorem 2. *In order that corresponding to* every *function $f(z)$ analytic and bounded in a region G there shall exist a sequence of polynomials $p_1(z), p_2(z), \cdots$ converging to $f(z)$ in G so that*

$$\varlimsup_{n\to\infty} \{\overline{\text{bound}}\ [\ |f(z) - p_n(z)|, z \text{ in } G]\} \leqq D(f, B) \tag{2}$$

it is necessary that G be finite and that $G + B$ be the complement of an infinite region. Moreover, if G is simply connected it is necessary that every point of the boundary of G be also a boundary point of the infinite region.

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[1] The usual definition of diameter is meant here, namely the maximum distance between any two points of the cluster set.

[2] See Seidel, "On the cluster values of analytic functions," *Transactions of the American Mathematical Society*, vol. 34 (1932), pp. 1-21.

2. Proof of Theorem 1. Let T be enclosed in a sequence of limited simply connected regions $T_1, T_2, \cdots$ all containing $T + J$ and such that every region T_{n+1} together with its boundary lies interior to T_n while no point exterior to T lies in all the T_n. Let $z = \alpha$ be a fixed point in T and let the function $\psi_n(z)$ map the region T_n conformally onto the region T so that the point $z = \alpha$ corresponds to itself and so that $\psi'_n(\alpha) > 0$. Then we have [3] uniformly in $T + J$

$$\lim_{n\to\infty} \psi_n(z) = z.$$

Let $p_n(z)$ denote a polynomial which approximates the function

$$f_n(z) = f[\psi_n(z)]$$

so that [4]

$$| p_n(z) - f_n(z) | \leqq 1/n, \qquad z \text{ in } T + J.$$

Thus we have for z in T, uniformly on any closed set in T,

$$\lim_{n\to\infty} p_n(z) = f(z).$$

We shall prove that these polynomials also fulfill the further requirements of the theorem. We find it convenient to do this in two steps as follows. We prove first that if z_0 be any point of J, if $z_1, z_2, \cdots$ be any sequence of points in G approaching z_0 as limit, and if $p_{\mu_1}(z), p_{\mu_2}(z), \cdots$ be any subsequence of the polynomials $p_n(z)$, then

$$\overline{\lim_{n\to\infty}} \, | f(z_n) - p_{\mu_n}(z_n) | \leqq d(f, z_0). \tag{3}$$

The second part of the proof is by way of contradiction, for we show that if the polynomials $p_n(z)$ do not fulfill (1) they violate (3).

We proceed with the proof of (3). Let z_0 be any point of J and let S_{z_0} denote the cluster set of $f(z)$ at z_0. If a positive ϵ be assigned, then for all n greater than a certain N we have simultaneously

(a) z_n and $\psi_{\mu_n}(z_n)$ both so near to z_0 that the image points $f(z_n)$ and $f_{\mu_n}(z_n)$ are both within distance ϵ of S_{z_0};

(b) $| p_{\mu_n}(z_n) - f_{\mu_n}(z_n) | \leqq \epsilon.$

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[3] See Walsh, "Interpolation and approximation by rational functions in the complex domain," *Colloquium Publications of the American Mathematical Society*, vol. 20 (1935), p. 35, Corollary 1.

[4] The function $f_n(z)$ is analytic in T_n and hence by Runge's theorem can be represented in T_n by a series of polynomials converging uniformly on any closed point set (such as $T + J$) lying wholly interior to T_n.

Consequently we have

$$|f(z_n) - p_{\mu_n}(z_n)| \leqq d(f, z_0) + 3\epsilon, \qquad n > N,$$

which proves (3).

Suppose now that (1) does not hold. Then there must exist a subsequence $p_{\lambda_1}(z), p_{\lambda_2}(z), \cdots$ of the $p_n(z)$ such that for EVERY $p_{\lambda_n}(z)$ we have

$$\overline{\text{bound}}\,[\,|f(z) - p_{\lambda_n}(z)|, z \text{ in } T] \geqq L > D(f, J).$$

Associate with each polynomial $p_{\lambda_n}(z)$ a point z_n in T within distance $1/n$ of J and at the same time so near to J that

$$|f(z_n) - p_{\lambda_n}(z_n)| > \overline{\text{bound}}\,[\,|f(z) - p_{\lambda_n}(z)|, z \text{ in } T] - 1/n.$$

Let $z'_1, z'_2, \cdots$ be a subsequence of the points z_n approaching a point z_0 on J as limit. Let $p_{\mu_n}(z)$ denote that polynomial of the sequence $p_{\lambda_n}(z)$ with which the point z'_n has been associated. Then for the sequence z'_n we have

$$\overline{\lim_{n\to\infty}}\,|f(z'_n) - p_{\mu_n}(z'_n)| \geqq L > D(f, J) \geqq d(f, z_0).$$

This contradicts (3) and completes the proof of the theorem.

3. Proof of Theorem 2. It follows at once from the boundedness of $f(z)$ in G that polynomial representation of $f(z)$ in G characterized by (2) requires that G be finite.

We note next that among all the functions $f(z)$ that are bounded in G are included all the functions that are analytic in $G + B$. Moreover, if $f(z)$ is analytic in $G + B$, then $D(f, B) = 0$; and any sequence of polynomials $p_n(z)$ which fulfills (1) thereby converges to $f(z)$ uniformly in $G + B$. But a necessary condition that every function analytic on a closed limited point set can be uniformly approximated thereon by a polynomial is that the point set be the complement of an infinite region.[5] In the present situation, where the point set in question consists of a region and its boundary, the condition that the point set be the complement of an infinite region is equivalent to the condition that B divide the extended plane into precisely two regions: G and an infinite region H.

When G is simply connected a further condition is necessary for the representation of a function $f(z)$ analytic and bounded in G by polynomials such that (2) holds. This condition is that B shall be the common boundary of G and H, or (what is the same) that the neighborhood of every point of B

[5] Walsh, *op. cit.*, p. 26, Theorem 17.

shall contain points of H. To prove this suppose there are points of B in the neighborhood of which there are no points of H. Choose for $f(z)$ any function which maps G conformally onto the interior of the unit circle. Let C denote the portion of B that bounds H and let R denote the finite region bounded by C. If polynomials $p_n(z)$ satisfy (2) while converging to $f(z)$ in G it follows that

$$\overline{\lim_{n\to\infty}}\,\{\mathrm{Max}\;[\,|\,p_n(z)\,|, z \text{ in } G+B]\} \leqq 1 + D(f, B) \leqq 3.$$

The $p_n(z)$ are therefore uniformly bounded on B and hence also on C and consequently in R as well. Thus they form a normal family in R from which we can extract a subsequence $p_{\lambda_n}(z)$ converging in R to a function $F(z)$ analytic in R and equal to $f(z)$ in G. If ζ be any point of B that lies in R, it follows that $|\,F(\zeta)\,| = 1$; also that $|\,F(z)\,| \leqq 1$ in the neighborhood of ζ. Then by the Principle of Maximum [6] $f(z)$ must be identically constant, which is impossible. Thus the mapping function $f(z)$ can not be represented in G by polynomials for which (2) holds. This completes the proof of Theorem 2.

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[6] See Walsh, *op. cit.*, p. 4.

INVARIANT MANIFOLDS NEAR AN INVARIANT POINT OF UNSTABLE TYPE.*

DANIEL C. LEWIS, JR.[1]

1. Introduction and formulation of the problem. In the case of real surface transformations, not necessarily analytic, the existence of an invariant curve in the neighborhood of an unstable invariant point (and containing this invariant point) was proved by Hadamard. In the case of the analytic surface transformations arising in dynamics, similar results were obtained by Poincaré and Birkhoff. Lattès [2] improved the results of Poincaré and also made some use of the methods of Hadamard. He obtained the generalization of Poincaré's result to transformations in three dimensional space. More recently, in the case of the surface transformations of dynamics, Birkhoff [3] has proved the existence of invariant curves of certain kinds near the given invariant point, but which do not contain this invariant point. In this paper we give a generalization of Hadamard's method to the case of transformations in spaces of an arbitrary number of dimensions. We shall, however, be interested primarily in analytic transformations, not because the method is inapplicable to the non-analytic case, but because the hypothesis of analyticity enables us to draw more complete conclusions. The main advantage in adopting Hadamard's method is that it gives a clearer picture of the way in which the invariant manifolds are the uniform limits of the successive images of other manifolds. We shall show that the method holds for complex spaces, and, since any transformation in a real space defined by power series can be interpreted as a trans-

* Received January 17, 1938.

[1] These results were obtained for the most part while the writer was a National Research Fellow.

[2] H. Poincaré, *Oeuvres de, publiées sous les auspices de l'académie des sciences*, vol. 1 (1928), Gauthier Villars, Paris, *Mémoire sur les courbes définies par les équations différentielles*, chapter XIX, especially pp. 202-203; J. Hadamard, "Sur l'itération et les solutions asymptotiques des équations différentielles," *Bulletin de la Société Mathématique de France*, vol. 29 (1901), pp. 224-228; G. D. Birkhoff, "Surface transformations and their dynamical applications," *Acta Mathematica*, vol. 43 (1922), pp. 1-119, especially pp. 45-49; S. Lattès, "Sur les équations fonctionelles qui definissent une courbe ou une surface invariante par une transformation," *Annali di Matematica*, ser. 3, vol. 13 (1907), pp. 1-137.

[3] G. D. Birkhoff, "Nouvelles recherches sur les systèmes dynamiques," *Memoriae, Pontificiae Academiae Scientiae Novi Lyncaei*, series 3, vol. 1, pp. 85-216, especially chapter III.

formation in the corresponding complex space (of twice the number of real dimensions), the analyticity of the invariant manifolds follows at once from well known theorems in the theory of analytic functions of several complex variables. This is because of the fact that these invariant manifolds may be taken as the uniform limits of sequences of analytic manifolds. Details will be given in § 5.

The proofs of Poincaré and Birkhoff use the theory of majorantes and thus also yield analyticity. Birkhoff's proof makes essential use of the hypothesis of "conservativity," a special property enjoyed by transformations arising in dynamics.

We consider a transformation T which with a suitable choice of coördinates can be written in the following form:

$$(1.1)\qquad \begin{aligned} \bar{x}_i &= s_i x_i + f_i(x, y), & (i = 1, 2, \cdots, m) \\ \bar{y}_h &= \sigma_h y_h + \phi_h(x, y), & (h = 1, 2, \cdots, n). \end{aligned}$$

It is supposed that f_i and ϕ_h are convergent power series in $x_1, \cdots, x_m, y_1, \cdots, y_n$ lacking constant and linear terms. As for the complex numbers s_i and σ_h, it is assumed that $|s_i| > |\sigma_h| > 0$. Denoting by s the minimum of the $|s_i|$ and by σ the maximum of the $|\sigma_h|$, we have $|s_i| \geqq s > \sigma \geqq |\sigma_h|$.

Under this transformation a sufficiently small neighborhood of the origin in the $m + n$ dimensional complex space is carried over into another neighborhood in a one-to-one manner, the origin itself being an invariant point. We agree once and for all to restrict attention to a neighborhood common to these two.

It is hardly necessary to note in passing that we are here dealing in a certain sense with the most general type of analytic transformation having a non-vanishing jacobian at an invariant point. In fact the matrix of the jacobian evaluated at the invariant point need only to have simple elementary divisors in order for the transformation to be representable in the form (1.1) in the neighborhood of the invariant point. The only other assumption is that the absolute values of the characteristic roots are not all equal to each other.

Let C be an arbitrary (local) complex "manifold" with the equations $y_h = y_h(x)$ $(h = 1, \cdots, n)$, where the $y_h(x)$ are single valued analytic functions of the x_i defined when each $|x_i|$ is sufficiently small and vanishing when $x_1 = \cdots = x_m = 0$. It is our principal aim to investigate the existence of a manifold C^* of this type every point of which sufficiently near the origin is transformed by T into a point of C^*. Such a manifold will be called invariant.

We shall find that such manifolds do exist if $s > 1$. If $s \leqq 1$, a consideration of the inverse transformation shows the existence of invariant mani-

folds of the type $x_i = x_i(y)$. Thus, invariant manifolds of some type always exist under our assumption, $s > \sigma$.

2. Preliminary considerations on implicit functions. We begin with the theory of implicit functions, proving a theorem which states that the implicit functions are defined not only in a neighborhood of a point but in a region which for our purposes is sufficiently extensive.

THEOREM 1. *Let $g_i(u,v)$ $(i = 1, \cdots, p)$ be p analytic functions of the $m + p$ complex variables, $u_1, \cdots, u_m, v_1, \cdots, v_p$ defined in a region R given by the inequalities,*

$$(2.1) \qquad |u_h| < A_h, \ |v_k| < B_k; \qquad (h = 1, \cdots, m;\ k = 1, \cdots, p).$$

Let the g_i furthermore satisfy the conditions

$$(2.2) \qquad g_i(0,0) = 0,$$

$$(2.3) \qquad \frac{\partial(g_1, \cdots, g_p)}{\partial(v_1, \cdots, v_p)} \neq 0 \ in\ R;$$

so that the functions F_{kh} defined by the equations

$$(2.4) \qquad \sum_{k=1}^{p} \frac{\partial g_i}{\partial v_k} F_{kh} + \frac{\partial g_i}{\partial u_h} = 0$$

must exist and be analytic in R. Moreover suppose that

$$(2.5) \qquad |F_{kh}| < M_{kh} \ in\ R$$

and let G_h $(h = 1, \cdots, m)$ be any m positive numbers such that

$$(2.6) \qquad \sum_{h=1}^{m} M_{kh} G_h \leqq B_k, \qquad G_h \leqq A_h.$$

Then there exists a unique set of analytic functions $v_i(u)$ $(i = 1, \cdots, p)$, of the m complex variables $u_1, \cdots, u_m$, defined in a region S given by

$$(2.7) \qquad |u_h| < G_h, \qquad (h = 1, \cdots, m),$$

such that each of the following three conditions is satisfied:

$$(2.8) \qquad |v_k(u)| < B_k \ for\ (u)\ in\ S, \qquad (k = 1, \cdots, p).$$

$$(2.9) \qquad v_k(0) = 0.$$

(2. 10) $\qquad g_k(u, v(u)) \equiv 0$ *in* S.[4]

Proof. It is readily verified by differentiation of (2. 4) and with the help of (2. 3), that the F's satisfy the identities

$$\sum_{a=1}^{p} \frac{\partial F_{kh}}{\partial v_a} F_{al} + \frac{\partial F_{kh}}{\partial u_l} \equiv \sum_{a=1}^{p} \frac{\partial F_{kl}}{\partial v_a} F_{ah} + \frac{\partial F_{kl}}{\partial u_h}, \qquad (h, l = 1, \cdots, m). \tag{2.11}$$

We now consider the system of differential equations

$$\frac{\partial v_k}{\partial u_h} = F_{kh}, \tag{2.12}$$

the complete integrability conditions of which are satisfied in virtue of (2. 11). Any set of analytic functions satisfying (2. 10) must be a solution of (2. 12), as we see by differentiating (2. 10) and using (2. 4). Conversely, it is readily verified that the unique solution of (2. 12) which vanishes at the origin will also satisfy (2. 10). We shall next show that this solution, which is known to exist in the neighborhood of the origin, may also be defined throughout S.

Let $(\bar{u})$ be an arbitrary point of S and let $u_h = u_h^0(t)$ be a straight line L joining the origin with $(\bar{u})$, considered as a point in the Euclidean space of $2m$ real dimensions. The real parameter t may be chosen as the distance of the point $(u^0(t))$ from the origin, so that we have $u_h^0(0) = 0$, $u_k^0(T) = \bar{u}_h$, $T \geqq 0$. Let the values of the functions $v_k^0(t)$ be determined from the existence theorems for ordinary differential equations by the following system:

$$\frac{dv_k^0}{dt} = \sum_{h=1}^{m} F_{kh}(u^0(t), v^0) \frac{du_h^0}{dt} \qquad (k = 1, \cdots, p) \tag{2.13}$$

together with the initial conditions $v_k^0(0) = 0$. The v^0's can be determined as functions of t so long as none of the $| v_k^0(t) |$ reaches the value B_k, beyond which the F's are not defined. Cf. (2. 1) and (2. 4). Now $| v_k^0(t) |$ can not increase any faster than it would if the right member of (2. 13) were replaced by $\sum_{h=1}^{m} M_{kh} \left| \frac{du_h^0}{dt} \right|$ from (2. 5). Integrating, we find, on using (2. 7) and

[4] It is not true, however, that the functions $v_i(u)$ necessarily yield all the solutions of the system of equations $g_i = 0$ in the region $| u_h | < G_h$, $| v_k | < B_k$, as we see from the following elementary example: Let $p = m = 1$ and let

$$g_1(u, v) = -1 + \exp 4\pi i (u_1 - v_1),$$

a function which is certainly both analytic and provided with a non-vanishing derivative with respect to v_1 in any finite region R we may choose to name, e. g. $| u_1 | < 2 = A_1$, $| v_1 | < 2 = B_1$. It turns out that $F_{11} \equiv 1$ and so we may choose $M_{11} = 2$ and $G = 1$. But in addition to the solutions provided by the theorem, viz. $v_1 = u_1$, the equation $g_1 = 0$ also admits (with infinitely many others) the solution, $u_1 = 0$, $v_1 = 1/2 < 2$.

(2.6) that we would have $|v_k^0(t)| \leqq \sum_{h=1}^{m} M_{kh} | u_h(t) | < B_k$ for $0 \leqq t \leqq T$. Here we have used the fact that L is a straight line and that $(u_h(t))$ is in S for $0 \leqq t \leqq T$. This shows that $v_k^0(t)$ is defined for $0 \leqq t \leqq T$. Now we set $v_k(\bar{u}) = v_k^0(T)$ and thus the v's are defined over S. We have shown at the same time that the v's as thus defined satisfy (2.8) and (2.9). That they also satisfy (2.12), and hence (2.10), and the fact that they are unique, now follows by an obvious adaptation from the classical theory of completely integrable total differential equations as presented by Morera or Mayer.[5]

It may be mentioned in passing that the above theorem is true also for real functions $g_i(u, v)$ of class C'' in the real variables (u) and (v), the solving functions $v_i(u)$ then being of class C'. The same proof is valid, if we understand L now as a straight line in a Euclidean space of m real dimensions.

3. On the transformation of a manifold belonging to K(α, β). We now return to the proper subject matter of the present investigation.

For greater brevity we use x and y without indicial subscripts to denote respectively the sets $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_n)$; and an x or y capitalized or tagged in some other manner will have a similar interpretation. For example $\bar{y}'$ would stand for the set $(\bar{y}'_1, \cdots, \bar{y}'_n)$. We admit, however, the following exceptions: $|x|$ will denote the largest of the numbers $|x_1|, \cdots, |x_m|$ and $|y|$ will denote the largest of the numbers $|y_1|, \cdots, |y_n|$. A further exception is introduced in § 4.

We use the notation d/dx_i to denote partial differentiation when the y's are functions of the x's. On the other hand $\partial/\partial x_i$ means that all the $m + n$ variables x and y are considered as independent. The symbol $|dy/dx|$ is used to represent the largest of the numbers $|dy_i/dx_h|$; $(i = 1, \cdots, n;\ h = 1, \cdots, m)$.

In considering any point on a manifold of the type of C, described in § 1, the x's will sometimes be called the abscissas and the y's the ordinates of the point in question.

Let us associate with any two positive numbers α and β the class $K(\alpha, \beta)$ of manifolds C such that $y(x)$ is defined and $|dy/dx| < \alpha$ for $|x| < \beta$.

Evidently $K(\alpha, \beta) \subset K(\alpha, \beta')$, if $\beta > \beta'$. We now prove

THEOREM 2. *β can always be chosen so small that any manifold C in $K(\alpha, \beta)$ is transformed by T into a manifold $\bar{C}$ in $K(\alpha, \beta s^*)$, where s^* is a preassigned positive number $< s$.*

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[5] Cf. Levi-Civita, *The Absolute Differential Calculus*, English translation by M. Long (1927), pp. 22-25.

Proof. The equations of $\bar{C}$ can be written with the parameters x in the form,

$$(3.1)\qquad \begin{aligned} \bar{x}_i &= s_i x_i + f_i(x, y(x)) \\ \bar{y}_h &= \sigma_h y_h(x) + \phi_h(x, y(x)). \end{aligned}$$

On elimination of x, these equations take the form $\bar{y}_h = \bar{y}_h(\bar{x})$, and it is our first purpose to show that $\bar{y}(\bar{x})$ is defined for $|\bar{x}| < \beta s^*$ if β is sufficiently small and C belongs to $K(\alpha, \beta)$. This we do by appealing to Theorem 1, taking $p = m$, $\bar{x}_i = u_i$, $x_i = v_i$, $g_i(u, v) \equiv s_i x_i - \bar{x}_i + f_i(x, y(x))$, $B_i = \beta$, $A_h = \infty$. For, since f_i does not contain linear terms in x and y and since $|dy/dx| < \alpha$, we see that all the derivatives of the $f_i(x, y(x))$ with respect to $x_1, \cdots, x_m$ approach zero with $|x|$ uniformly with respect to the manifolds C considered. It follows that for $C \subset K(\alpha, \beta)$ we may take the M_{kh} of Theorem 1 equal to an expression of the form $\delta_{kh} |s_h^{-1}| + \epsilon$, where $\delta_{kh} = 0$ if $k \neq h$ and $\delta_{hh} = 1$, and where ϵ tends uniformly to O with β. Hence by taking β sufficiently small, we see from (2.6) that we may take $G_h = \beta s^*$. All this shows that we may uniquely and analytically solve the first m of equations (3.1) for the x's in terms of the $\bar{x}$'s by means of equations of the form $x_i = x_i(\bar{x})$, where the $x_i(\bar{x})$ are defined for $|\bar{x}| < \beta s^*$, and in such manner that $x_i(0) = 0$ and $|x(\bar{x})| < \beta$. In virtue of these last inequalities we can substitute the values of x_i into the last n of equations (3.1) and thus obtain the required equations of $\bar{C}$ in the form $\bar{y}_h = \bar{y}_h(\bar{x})$ defined for $|x| < \beta s^*$, provided again that β is sufficiently small. And the result holds for any C in $K(\alpha, \beta)$.

We next prove that for the functions thus obtained $|d\bar{y}/d\bar{x}| < \alpha$ for $|\bar{x}| < \beta s^*$ and $C \subset K(\alpha, \beta)$, at least for β sufficiently small, i. e. perhaps still smaller than the value already assigned to β. Differentiation of the equations $\bar{y}_h = \bar{y}_h(\bar{x})$ with respect to x_i yields

$$\frac{d\bar{y}_h}{dx_i} = \sum_{k=1}^{m} \frac{d\bar{y}_h}{d\bar{x}_k} \frac{d\bar{x}_k}{dx_i}.$$

Solving these equations for the derivative $d\bar{y}_h/d\bar{x}_k$, we get

$$\frac{d\bar{y}_h}{d\bar{x}_k} = \frac{N}{D},$$

where D is the functional determinant $\dfrac{d(\bar{x}_1, \cdots, \bar{x}_m)}{d(x_1, \cdots, x_m)}$ and N is the determinant obtained by replacing the element $d\bar{x}_k/dx_i$ in D by $d\bar{y}_h/dx_i$ for each i. Again using the fact that f and ϕ contain no linear terms and that $|dy/dx| < \alpha$, we see from differentiation of (3.1) that

$$\frac{d\bar{x}_k}{dx_i} = s_k \delta_{ki} + \frac{\partial f_k}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial f_k}{\partial y_j} \frac{dy_j}{dx_i} = s_k \delta_{ki} + \xi_{ki}$$

$$\frac{d\bar{y}_h}{dx_i} = \sigma_h \frac{dy_h}{dx_i} + \frac{\partial \phi_h}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial \phi_h}{\partial y_j} \frac{dy_j}{dx_i} = \sigma_h \frac{dy_h}{dx_i} + \eta_{hi}$$

where the ξ's and η's approach zero with $| x |$ uniformly as to C. We thus find

$$\frac{d\bar{y}_h}{d\bar{x}_k} = \frac{N}{D} = \frac{\sigma_h}{s_k} \frac{dy_h}{dx_k} + \theta_{hk},$$

where θ also tends uniformly to zero. Since, however, $| \sigma_h | \leqq \sigma < s \leqq | s_k |$, it follows that

$$\left| \frac{d\bar{y}_h}{d\bar{x}_k} \right| \leqq \left| \frac{\sigma_h}{s_k} \right| \cdot \alpha + | \theta_{hk} | < \frac{\sigma}{s} \alpha + | \theta_{hk} | < \alpha,$$

if β is sufficiently small. This completes the proof.

If $s > 1$, we may take $s^* = 1$ and we have the following corollary to Theorem 1:

COROLLARY. *If $s > 1$ we may always take β so small that any manifold C of $K(\alpha, \beta)$ is transformed by T and its iterates into manifolds of $K(\alpha, \beta)$.*

4. On the transformation of two manifolds. For the sake of brevity we shall hereafter, in reference to an expression a_i, depending on an index i, use the notation $\| a \| = \Sigma \, | a_i |$ where the summation is extended over the range of the index, which as a matter of fact will always be either $1, 2, \cdots, m$ or $1, 2, \cdots, n$.

Let C and C' be any two manifolds in $K(\alpha, \beta)$. If we consider two points with the same abscissas, one of them (x, y) on C and the other (x, y') on C', all the differences $y_h - y'_h$ will vanish for $x = 0$. Hence it is easily seen that the quantity $\| y - y' \| / \| x \|$ will have in a fixed neighborhood D of the origin a certain least upper bound $\omega \leqq 2\alpha n$. Now let $\bar{C}$ and $\bar{C}'$ denote the transforms of C and C' respectively. Denote by $\bar{\omega}$ the quantity analogous to ω determined from $\bar{C}$ and $\bar{C}'$. Then we may state as follows our

THEOREM 3. *The ratio $\bar{\omega}/\omega$ does not exceed a fixed number $\rho < 1$, if the domain D is chosen sufficiently small. For ρ we may take any real number greater than σ/s and less than 1. D may be chosen independently of C and C' if β is sufficiently small.*

Proof. Denote by Y and Y' the ordinates of C and C' respectively corresponding to the same abscissas X; $\bar{y}$ and $\bar{y}'$ the analogous ordinates of $\bar{C}$ and $\bar{C}'$. The points $(X, \bar{y})$ and $X, \bar{y}')$ will be the transforms of two points (x, y)

and (x', y') situated on C and C' respectively. From the hypothesis that $C \subset K(\alpha, \beta)$ we have

$$(4.1) \qquad |y| \leqq \max \left| \frac{dy}{dx} \right| \cdot \| x \| \leqq \alpha \| x \|, \text{ i.e. } \| y \| \leqq n\alpha \| x \|$$

and from the hypothesis that the f's and ϕ's begin with terms of the second degree we get

$$| X_i - s_i x_i | \leqq \eta(\| x \| + \| y \|) \leqq \eta(1 + n\alpha) \| x \|$$

where

$$\eta \left(\geqq \left| \frac{\partial f_i}{\partial x_j} \right|, \left| \frac{\partial f_i}{\partial y_j} \right|, \left| \frac{\partial \phi_i}{\partial x_j} \right|, \left| \frac{\partial \phi_i}{\partial y_j} \right| \right)$$

can be taken arbitrarily small by taking the dimensions of D sufficiently small. This inequality gives us

$$s | x_i | \leqq | s_i x_i | \leqq | s_i x_i - X_i | + | X_i | \leqq \eta(1 + n\alpha) \| x \| + | X_i |,$$

and hence

$$s \| x \| \leqq \eta m (1 + n\alpha) \| x \| + \| X \|$$

and thus we obtain

$$\| x \| \leqq \frac{\| X \|}{s - \eta m (1 + n\alpha)}.$$

If (x, y'_0) is the point of C' corresponding to the abscissas x, we have

$$(4.2) \qquad \| y - y'_0 \| \leqq \omega \| x \| \leqq \frac{\omega \| X \|}{s - \eta m (1 + n\alpha)}$$

$$(4.3) \qquad | y'_h - y'_{0h} | \leqq \max \left| \frac{dy'}{dx} \right| \cdot \| x' - x \| \leqq \alpha \| x' - x \|.$$

On the other hand by the very definition of X, x, x', y, y', we have

$$X_i = s_i x_i + f_i(x, y) \quad \text{and} \quad X_i = s_i x'_i + f_i(x', y');$$

and hence

$$s_i(x_i - x'_i) = f_i(x', y') - f_i(x, y).$$

Since the first derivatives of f_i and ϕ_i are all in absolute value less than η, we can write

$$s | x_i - x'_i | \leqq | s_i(x_i - x'_i) | \leqq \eta \| x - x' \| + \eta \| y - y' \|$$
$$s \| x - x' \| \leqq m\eta \| x - x' \| + m\eta \| y - y' \|$$

$$(4.4) \qquad \| x - x' \| \leqq \frac{m\eta \| y - y' \|}{s - m\eta}.$$

From (4.3) and (4.4) we obtain

$$| y'_h - y'_{0h} | \leqq \frac{\alpha m\eta \, \| y - y' \|}{s - m\eta}$$

$$| y'_h - y_h | \leqq | y'_h - y'_{0h} | + | y'_{0h} - y_h | \leqq \frac{\alpha m\eta \, \| y - y' \|}{s - m\eta} + | y'_{0h} - y_h |.$$

Therefore it follows (summing over h) that

$$\| y' - y \| \leqq \| y'_0 - y \| / \left(1 - \frac{\alpha n m\eta}{s - m\eta}\right) = \frac{(s - m\eta) \| y'_0 - y \|}{s - m\eta(1 + \alpha n)}.$$

From (4.2) we now obtain immediately

$$\| y' - y \| \leqq \frac{(s - m\eta)\omega}{[s - m\eta(1 + n\alpha)]^2} \| X \|. \tag{4.5}$$

By definition, we have $\bar{y}_h = \sigma_h y_h + \phi_h(x, y)$ and $\bar{y}'_h = \sigma_h y'_h + \phi_h(x', y')$. Hence

$$\begin{aligned} | \bar{y}_h - \bar{y}'_h | &\leqq | \bar{y}_h - \bar{y}'_h - \sigma_h(y_h - y'_h) | + | \sigma_h | \cdot | y_h - y'_h | \\ &= | \phi_h(x, y) - \phi_h(x', y') | + | \sigma_h | \cdot | y_h - y'_h | \\ &\leqq \eta(\| x - x' \| + \| y - y' \|) + | \sigma | \cdot | y_h - y'_h |. \end{aligned}$$

Hence

$$\| \bar{y} - \bar{y}' \| \leqq n\eta \| x - x' \| + (n\eta + \sigma) \| y - y' \|.$$

From (4.4) we obtain

$$\| \bar{y} - \bar{y}' \| \leqq \left[\frac{mn\eta^2}{s - m\eta} + n\eta + \sigma\right] \| y - y' \|,$$

while from (4.5) we see further that

$$\begin{aligned} \| \bar{y} - \bar{y}' \| &\leqq \left[\frac{mn\eta^2}{s - m\eta} + n\eta + \sigma\right] \frac{(s - m\eta)\omega}{[s - m\eta(1 + n\alpha)]^2} \| X \| \\ &= \left[\frac{\sigma}{s} + \xi(\eta)\right] \omega \| X \|, \end{aligned} \tag{4.6}$$

where $\xi(\eta)$ is real and tends to zero with η and therefore also with the dimensions of D. Since $\sigma < s$, we have $0 < \sigma/s + \xi(\eta) < \rho < 1$, provided that D is sufficiently small. Thus we have proved the desired result, namely, that $\bar{\omega}$, the least upper bound of $\| \bar{y} - \bar{y}' \| / \| X \|$ can not exceed $\rho\omega$.

The fact that D may be chosen independently of the particular manifolds C and C' considered, so long as they both belong to $K(\alpha, \beta)$ and β is first chosen sufficiently small, follows from two considerations: In the first place, according to Theorem 2, C, C', $\bar{C}$, $\bar{C}'$ have a common domain of definition independent of C and C', namely, the smaller of the two domains $| x | < \beta$ and $| x | < \beta s^*$. In the second place, the quantity $\xi(\eta)$ in (4.6) is independent of C and C'. As a matter of fact it is not difficult to see that by taking β

still smaller, if necessary, we may choose D to be the smaller of the two domains $|x| < \beta$ and $|x| < \beta s^*$. But this result is unnecessary for our purposes.

5. The invariant manifolds for $s > 1$. Let C_0 denote an arbitrary manifold in $K(\alpha, \beta)$ with the equations $y = y^{(0)}(x)$. We denote by C_k the k-th transform of C_0 under T. That is, C_0 is transformed into C_1 by T; C_1 is transformed into C_2 and so forth. In order to ensure that all the C_k also belong to $K(\alpha, \beta)$, we add the hypothesis concerning T that $s > 1$ and take β sufficiently small as described in the corollary of Theorem 2. Let D be the domain referred to in Theorem 3, dependent on the class $K(\alpha, \beta)$ but independent of the particular C and C' belonging to this class. It is necessarily contained in the region $|x| < \beta$. Let the equations of C_k be $y = y^{(k)}(x)$. We now prove our main result,

THEOREM 4. *There exists an invariant analytic manifold C^* with the equations $y = y^*(x)$, such that $\lim_{k=\infty} y^{(k)}(x) = y^*(x)$ uniformly in D.*

Proof. Let ω_k be the least upper bound of $\| y^{(k)} - y^{(k-1)} \| / \| x \|$ in D. By Theorem 3, we have $\omega_{k+1} \leqq \rho\omega_k$. It follows by induction that $\omega_k \leqq \rho^{k-1}\omega_1$. Thus we have

$$| y_i^{(k)}(x) - y_i^{(k-1)}(x) | \leqq \| y^{(k)}(x) - y^{(k-1)}(x) \| \leqq \omega_k \| x \| \leqq \rho^{k-1}\omega_1 m\beta,$$

and, since the series $\sum_{k=1}^{\infty} \rho^{k-1}\omega_1 m\beta$ is convergent, ρ being < 1, the uniform convergence of the sequence $y_i^{(0)}, y_i^{(1)}, y_i^{(2)}, \cdots$ follows from the classical theorem of Weierstrass on uniform convergence.

It follows from another theorem of Weierstrass [6] that C^* is analytic in the interior of D, since D is a domain of m complex dimensions or $2m$ real dimensions.

Furthermore C^* is invariant. For $\lim_{k=\infty} C_k = C^*$ while $\lim_{k=\infty} C_{k+1} =$ the transform of C^*, since T is continuous. It follows that C^* is identical with its transform.

THEOREM 5. *C^* is independent of the initially chosen manifold C_0.*

Proof. Let C'_k be another sequence of manifolds of the same type as C_k, i. e. each C'_k belongs to $K(\alpha, \beta)$; T carries C'_k into C'_{k+1}. Let $y = y^{(k)\prime}(x)$ be the equations of C'_k. Then, if ω is the least upper bound of

$$\| y^{(1)} - y^{(1)\prime} \| / \| x \|$$

in D, we find from Theorem 3 that

[6] Cf. W. F. Osgood, *Lehrbuch der Funktionentheorie*, vol. 2 (1924), first section, p. 13, 1. Satz.

$$| y_i^{(k)} - y_i^{(k)\prime} | \leqq \| y^{(k)} - y^{(k)\prime} \| \leqq \rho^k \omega \| x \| \leqq \rho^k \omega m \beta.$$

And, since $\rho < 1$, it follows that $\lim_{k=\infty} C'_k = \lim_{k=\infty} C_k$.

COROLLARY. *If C belongs to $K(\alpha, \beta)$ and is invariant it must necessarily coincide with the manifold C^* of Theorems 4 and 5 throughout D.*

Proof. Referring to the proof of Theorem 5, let us take $C'_0 = C$. Then, since C is assumed to be invariant, we have $C'_k = C$, and hence $C = \lim_{k=\infty} C_k = C^*$, throughout D where C^* is defined.

We might add the obvious remark that C^* can be continued analytically over the complete domain of definition of C and the two manifolds will then coincide throughout.

6. Note on real transformations. So far we have looked at the transformation T from the point of view of the complex domain. If, however, we are primarily interested in a real transformation and real analytic manifolds, we obtain similar results.

If T is a real transformation, we do not necessarily mean that all the coefficients in the power series appearing in (1.1) are real nor that real values for x and y represent real points. For, in order to reduce the linear terms of the transformation to the canonical form displayed in (1.1), it might be necessary to introduce conjugate imaginary variables, in which case the coefficients would not all be real and real values of the variables would not correspond to real points in the transformation. Since conjugate imaginary variables are associated with conjugate imaginary characteristic roots and since the characteristic roots s_i are all in absolute value greater than the characteristic roots σ_h, it is clear that, if any such conjugate imaginary pairs occur, none of the variables x can be paired with a variable y.

We briefly sketch the type of argument to be used in connection with the real theory. The C_0 of the previous paragraph may be taken in such a way that values of x corresponding to real points in the m-dimensional x-space will yield real points in the n-dimensional y-space. Then $C_1, C_2, \cdots$ will automatically have the same property, if T is real. The manifolds C_k are known to converge uniformly in a complex domain in which our real domain is imbedded. Hence our real manifolds will converge uniformly to a manifold which is necessarily both real and analytic.

It is also possible to use the method of this paper to treat directly real transformations which are not necessarily analytic, as, in fact, was done originally by Hadamard (*loc. cit.*) for the case $n = m = 1$. The invariant manifolds will then, of course, not necessarily be analytic.

CORNELL UNIVERSITY.

ON STRONG SUMMABILITY.*

By H. J. Hamilton and J. D. Hill.

1. **Definitions.** Being given in the complex domain a matrix (a_{mk}) $(k, m = 1, 2, 3, \cdots)$, and a positive real number p we shall say (following Szász, *Brown University Lectures*, 1934-35) that a complex sequence $\{s_k\}$ is *strongly summable of order p to the complex number s by (a_{mk})*, or simply *strongly summable* to s, if for each $m > 0$ the series $\sigma_m \equiv \sum_k a_{mk} \mid s - s_k \mid^p$ is convergent, and if $\lim_m \sigma_m = 0$. If each convergent $\{s_k\}$ is strongly summable to its limit, then the matrix (a_{mk}) will be called *strongly regular*. The terms *summable* and *regular*, unmodified, are to be interpreted in the usual Silverman-Toeplitz sense.

We shall say that a matrix is *of type U* if it has the property that whenever any sequence is strongly summable to both s and t, we are permitted to conclude that $s = t$. The phrase *of type U for convergent sequences* will mean that the strong summability of any *convergent* sequence to both s and t implies $s = t$; and analogously for bounded sequences.

2. **Strong regularity.** To find necessary and sufficient conditions for strong regularity it is convenient to introduce three lemmas. We mention in passing that the index p plays no essential rôle here.

Lemma 1. *In order that (a_{mk}) be strongly regular it is necessary and sufficient that it be regular for real non-negative null sequences.*

Lemma 2. *In order that (a_{mk}) be regular for real non-negative null sequences it is necessary and sufficient that it be regular in the space (c_0) of real null sequences.*

Lemma 3. *In order that (a_{mk}) be regular in (c_0) it is necessary and sufficient that*

$$(1) \qquad \lim_m a_{mk} = 0 \qquad (k = 1, 2, \cdots),$$

$$(2) \qquad \sup_m \sum_k \mid a_{mk} \mid < \infty.$$

* Received January 24, 1938; Revised April 25, 1938.

Proofs. The first is obvious, and the second follows from the fact that an arbitrary real null sequence may be written as the difference of two real non-negative null sequences. To prove the third observe first that in case (a_{mk}) is real, a slight modification of the regularity proof for convergent sequences [1] suffices to establish (1) and (2) as necessary and sufficient.

In the general case we may write $a_{mk} \equiv a'_{mk} + ia''_{mk}$ where a'_{mk}, a''_{mk} are real. It is clear that (a_{mk}) is regular in (c_0) if and only if each of the matrices (a'_{mk}), (a''_{mk}) has the same property. In order that the latter be true one easily shows, by the above statement for real matrices, that (1) and (2) are necessary and sufficient, and the proof is complete.

From these three lemmas we obtain the following theorem, in answer to a question raised by Dr. Szász.

THEOREM 1. *In order that (a_{mk}) be strongly regular the conditions* (1) *and* (2) *are necessary and sufficient.*

Since the Silverman-Toeplitz conditions include (1) and (2) we observe that every regular matrix is strongly regular.

3. Matrices of type U. To treat the question of uniqueness of the strong limit we need the following lemma.

LEMMA 4. *For each $p > 0$ there exists a constant K, depending on p, such that for all values of a and b we have $|a + b|^p \leqq K(|a|^p + |b|^p)$.*

Proof. We may assume that $a + b \neq 0$ and hence that a, say, is not zero. Then we have

$$\frac{|a+b|^p}{|a|^p + |b|^p} \leqq \frac{(|a| + |b|)^p}{|a|^p + |b|^p} = \frac{(1 + |b/a|)^p}{1 + |b/a|^p}.$$

Now it is easy to verify that if $0 < p \leqq 1$, $F(x) \equiv (1 + x)^p/(1 + x^p) \leqq 1$ for $0 \leqq x < \infty$, and if $p > 1$, $F(x) \leqq 2^{p-1}$ for $0 \leqq x < \infty$. Consequently, the result follows with K equal to 1 in the first case and 2^{p-1} in the second.

It will be useful to investigate first of all some of the properties of matrices for which a constant $M \geqq 0$ exists such that

$$a_{mk} \geqq 0 \text{ for each } m > 0 \text{ and all } k > M, \tag{3}$$

$$\lim_m a_{mk} = 0 \quad \text{for} \quad 1 \leqq k \leqq M. \tag{4}$$

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[1] See, for example, Banach, *Théorie des opérations linéaires*, pp. 90-91.

For such matrices the strong limit has the property of finite additivity as shown by the following theorem.

THEOREM 2. *If $\{s_k\}$ and $\{t_k\}$ are strongly summable to s and t, respectively, by a matrix (a_{mk}) which satisfies* (3) *and* (4), *then for all values of a and b the sequence $\{as_k + bt_k\}$ is strongly summable to $as + bt$.*

Proof. Using Lemma 4 we may establish the inequality

$$| \sum_k a_{mk} | (as_k + bt_k) - (as + bt) |^p | \leqq \sum_{k \leqq M} | a_{mk} | \; | a(s_k - s) + b(t_k - t) |^p$$
$$+ K | a |^p \sum_{k > M} a_{mk} | s_k - s |^p + K | b |^p \sum_{k > M} a_{mk} | t_k - t |^p$$

for each $m > 0$, and the conclusion is apparent.

THEOREM 3. *If (a_{mk}) satisfies* (3), (4), *and if*

(5) $$\sum_k a_{mk} \text{ is convergent for each } m > 0,$$

then the condition

(6) $$\overline{\lim_m} \, | \sum_k a_{mk} | > 0$$

is necessary and sufficient that (a_{mk}) be of type U.

Proof. If (a_{mk}) is of type U the sequence $1, 1, 1, \cdots$, which is strongly summable to 1, can not be strongly summable to 0; hence the necessity. On the other hand, if $\{s_k\}$ is strongly summable to s and also to t, then by Theorem 2 the sequence $\{(1)s_k + (-1)s_k\} = 0, 0, 0, \cdots$ is strongly summable to $s - t$. Thus we must have $\lim_m | s - t |^p \sum_k a_{mk} = 0$, and if (6) holds this is possible only if $s = t$.

THEOREM 4. *If $\{s_k\}$ is strongly summable to s by a matrix (a_{mk}) which satisfies* (3), (4), (5), (6), *then s must be a limit point of $\{s_k\}$.*

Proof. If we set $\sigma'_m \equiv \sum_{k \leqq M} a_{mk} | s_k - s |^p$, $\sigma''_m \equiv \sum_{k > M} a_{mk} | s_k - s |^p$ then $\sigma'_m + \sigma''_m \to 0$, whence, by (4), $\sigma''_m \to 0$. But the latter is impossible if s is not a limit point of $\{s_k\}$. For in that event a number $d > 0$ exists such that $| s_k - s |^p > d$ for all k. Then $\sigma''_m > d \sum_{k > M} a_{mk}$, and since by (4) and (6), $\overline{\lim_m} \, | \sum_k a_{mk} | = \overline{\lim_m} \sum_{k > M} a_{mk} > 0$, we have $\overline{\lim_m} \, \sigma''_m > 0$.

The foregoing property of strong summability may be used in obtaining the following comparison with ordinary summability.

THEOREM 5. *Suppose that* (a_{mk}) *satisfies* (3), (4), *and*

(7) $$\lim_m \sum_k a_{mk} = 1.$$

Then if $\{s_k\}$ *is strongly summable of order* 1 *to* s *by* (a_{mk}), *it is also summable to* s *by the same matrix. Moreover, the converse is false for every order* p.

Proof. From the inequality

$$\left| \sum_k a_{mk}(s - s_k) \right| \leqq \sum_{k \leqq M} |\, a_{mk} \,|\, |\, s - s_k \,| + \sum_{k > M} a_{mk} \,|\, s - s_k \,|$$

we see that $\sum_k a_{mk}(s - s_k) = o(1)$. Consequently we obtain

$$\sum_k a_{mk} s_k = s \sum_k a_{mk} + o(1),$$

and the first part of the conclusion follows.

Since the sequence $\{(-1)^{k-1}\}$ is $(C, 1)$ summable to 0, the falsity of the converse is apparent from Theorem 4.

Proceeding to the general case we begin by establishing the following result.

THEOREM 6. *In order that a strongly regular matrix* (a_{mk}) *be of type U for convergent sequences, the condition* (6) *is necessary and sufficient.*

Proof. The necessity follows as in the proof of Theorem 3. To establish the sufficiency let $\{s_k\}$ be an arbitrary convergent sequence. By the assumed regularity we have

(a) $$\lim_m \sum_k a_{mk} \,|\, s - s_k \,|^p = 0,$$

where $s \equiv \lim_k s_k$. Denote by t any number for which

(b) $$\lim_m \sum_k a_{mk} \,|\, t - s_k \,|^p = 0.$$

Setting $u_k \equiv |\, s - s_k \,|^p + |\, t - s_k \,|^p$ we obtain

(c) $$\lim_k u_k = |\, t - s \,|^p,$$

and from (a), (b),

(d) $$\lim_m \sum_k a_{mk} u_k = 0.$$

Now from (6) we infer the existence of a sequence $\{m_n\}$ tending to infinity, and of a number $\lambda \neq 0$, such that

(e) $$\lim_n \sum_k \lambda a_{m_n k} = 1.$$

Since the matrix $(b_{nk}) \equiv (\lambda a_{m_nk})$ satisfies (1), (2), (e), it is regular in the Silverman-Toeplitz sense. Consequently, from (c) and (d) we find that $\lim_n \sum_k b_{nk}u_k = | t - s |^p = 0$, whence $t = s$.

At this point the question naturally arises whether or not conditions (1), (2), (6) are sufficient to define the strong limit uniquely in general, and the following example shows that they are insufficient even for the case of real matrices and bounded sequences.

Example 1. Define $a_{m,3n-2} = 1/m$ $(m \geqq 3n - 2)$, $a_{m,3n-1} = 1/m$ $(m \geqq 3n - 1)$, $a_{m,3n} = -1/m$ $(m \geqq 3n)$ for $n = 1, 2, 3, \cdots$, and let $a_{mk} = 0$ otherwise. Then (a_{mk}) is strongly regular, and condition (6) is satisfied. But if we let $s_{3n-2} = 1$, $s_{3n-1} = 0$, $s_{3n} = -1$ for $n = 1, 2, 3, \cdots$, one easily shows that the sequence $\{s_k\}$ is strongly summable by (a_{mk}) to both 0 and 2.

It may be of interest to point out that this example suffices also to show that the conclusion of Theorem 4 is no longer valid in the absence from the hypotheses of condition (3).

In view of Example 1 it becomes necessary to seek further conditions on (a_{mk}) in order to insure uniqueness for strongly summable sequences that are not necessarily convergent. The problem of finding such conditions which are necessary as well as sufficient remains unsolved. Certain sufficient conditions, however, are introduced in the following lemmas and theorems.

LEMMA 5. *If, for any subsequence $\{m_n\}$ of $\{m\}$, (a_{m_nk}) is of type U, then (a_{mk}) is of type U.*

LEMMA 6. *If, for any bounded sequence of complex numbers $\{c_m\}$, $(c_m a_{mk})$ is of type U, then (a_{mk}) is of type U.*

LEMMA 7.[2] *If either (a'_{mk}) or (a''_{mk}) is of type U, then (a_{mk}) is of type U.*

The proofs of these lemmas are immediate consequences of the definition of matrices of type U.

THEOREM 7. *In order that a given matrix (a_{mk}) be of type U in general, it is sufficient that conditions* (4), (5), (6), *and the following condition* (8) *should be satisfied.*

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[2] The authors are indebted to Dr. Otto Szász for suggesting the introduction of Lemma 7, and also for pointing out the generalization of Theorems 7 and 8 obtained by replacing the condition $\gamma = \pi/4$ of the original manuscript by the condition $\gamma < \pi/2$.

(8) *There exist a value* ϕ_{mk} *of* $\arg a_{mk}$ $(m, k = 1, 2, 3, \cdots)$, *a sequence* $\{\phi_m\}$, *a positive constant* $\gamma < \pi/2$, *and a constant* $M \geqq 0$, *such that, for each* $m > 0$ *and for all* $k > M$ *we have* $\phi_m - \gamma \leqq \phi_{mk} \leqq \phi_m + \gamma$.

Remark. Stated in geometric language, condition (8) postulates the existence in the complex plane of congruent sectors Q_m of angle less than π, such that for each m $\arg a_{mk}$ lies in Q_m for all $k > M$.

Proof. Setting $e^{-i\phi_m}a_{mk} \equiv \alpha_{mk} + i\beta_{mk}$, where α_{mk} and β_{mk} are real, we see by Lemmas 7 and 6 that it will suffice to prove that (α_{mk}) is of type U. Now $\alpha_{mk} = |a_{mk}| \cos(\phi_{mk} - \phi_m) \geqq |a_{mk}| \cos\gamma \geqq 0$ for each $m > 0$ and all $k > M$, so that (α_{mk}) satisfies (3). That (α_{mk}) satisfies (4) and (5) is clear. Moreover, from the inequality $|\beta_{mk}| \leqq \alpha_{mk} \tan\gamma$ $(m > 0,\ k > M)$, it follows that

$$\Big|\sum_{k>M} a_{mk}\Big| = \Big|\sum_{k>M} e^{-i\phi_m}a_{mk}\Big| \leqq \sum_{k>M} \alpha_{mk} + \sum_{k>M} |\beta_{mk}| \leqq (1 + \tan\gamma) \sum_{k>M} \alpha_{mk}$$

for all m, whence, in view of (4) and (6), (α_{mk}) satisfies (6). The conclusion now follows from Theorem 3.

The question of whether any such "sector condition" as (8) is necessary for uniqueness is answered in the negative by the next example.

Example 2. Define $a_{mm} = a_{m,m-2} = 1$, $a_{m,m-1} = -1$ for $m \geqq 3$, and $a_{mk} = 0$ otherwise. Then (a_{mk}) satisfies (1), (2), and (6), but not (8). It is nevertheless of type U. For supposing that $\{s_k\}$ is strongly summable to s, we have, for $m \geqq 3$,

$$\sigma_m \equiv \sum_k a_{mk} |s - s_k|^p = |s - s_{m-2}|^p - |s - s_{m-1}|^p + |s - s_m|^p \to 0,$$

whence

$$\sigma_m + \sigma_{m+1} + \sigma_{m+2} = |s - s_{m-2}|^p + |s - s_m|^p + |s - s_{m+2}|^p \to 0,$$

so that $s_m \to s$. Thus (a_{mk}) sums convergent sequences only, and, indeed, only to their proper limits.

THEOREM 8. *In order that a given matrix* (a_{mk}) *be of type* U *for bounded sequences, it is sufficient that conditions* (5), (6), *and the following condition* (9) *should be satisfied.*

(9) *There exist a value* ϕ_{mk} *of* $\arg a_{mk}$ $(m, k = 1, 2, 3, \cdots)$, *sequences* $\{\phi_m\}$, $\{k_n{}^m\}$, *and a positive constant* $\gamma < \pi/2$ *such that* (i) *for each* $m > 0$ *and for all* $k \neq k_n{}^m$ *we have* $\phi_m - \gamma \leqq \phi_{mk} \leqq \phi_m + \gamma$, *and* (ii) $\lim_m \sum_n |a_{mk_n{}^m}| = 0$.

Remark. It is clear that any matrix which satisfies (4) and (8) also satisfies (9), whereas the converse is not true.

Proof. For each $m > 0$ define b_{mk} as 0 if $k = k_n{}^m$, and as a_{mk} if $k \neq k_n{}^m$ $(k, n = 1, 2, 3, \cdots)$. It is clear that it will suffice to prove that (b_{mk}) is of type U; but this is immediate, since (b_{mk}) satisfies the conditions of Theorem 7, with $M = 0$.

Theorems 7 and 8 can be extended by application of Lemma 5 in an obvious manner. (It may be noted that the condition $\underline{\lim}_m \mid \sum_n a_{mk} \mid > 0$ insures the satisfaction of (6) for all subsequences $\{m_n\}$—and of course conversely.) That no such "sector condition" as (8) or (9) is necessary for uniqueness of the strong limit in any case, is evident from Example 2 above.

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INVARIANTS DERIVED FROM LOOPING COEFFICIENTS.*

By E. R. van Kampen.

A looping coefficient[1] was first introduced by Brouwer for two non-intersecting manifolds of complementary dimensions k and $n-k-1$ embedded in n-dimensional space. Let the product of the two manifolds be transformed into the unit sphere of n-space by means of the vectors from points of the first manifold to points of the second manifold. Then Brouwer defines the looping coefficient of the two manifolds as the degree of the resulting transformation. It is essential for this definition that the manifolds are embedded in ordinary n-space, but if the manifolds are replaced by cycles of the same dimensions, the definition retains its meaning. When the identity of the looping coefficient of two non-interesting cycles of complementary dimensions with the intersection number of one of the cycles and any chain bounded by the other cycle was established, the way was paved for the introduction of a looping coefficient for cycles which are homologous to zero in any arbitrary orientable n-dimensional manifold M_n. At this stage the looping coefficients still were characteristics of the cycles only and did not contain any invariant property of the manifold M_n itself.

The next step was the introduction of looping coefficients also for so-called zero divisors of M_n, that means for cycles in M_n of which a certain integral multiple is homologous to zero in M_n. The definition was now formulated as follows: If α times the cycle a_1 is the boundary of the chain c and the cycle a_2 of complementary dimension to a_1, is a zero-divisor and does not meet a_2, then the looping coefficient $L(a_1, a_2)$ is the intersection number of c and a_2 divided by α. The condition that a_2 is a zero-divisor is necessary and sufficient for $L(a_1, a_2)$ to be independent of the particular choice of c. In this last form the looping coefficients are a priori capable of carrying invariant properties of the manifold M_n, since it turns out that $L(a_1, a_2)$ is determined modulo 1 by the homology classes of the cycles a_1 and a_2. As a consequence of (6) (cf. **1**), the looping coefficients of zero-divisors do not carry

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[1] L. E. J. Brouwer, "On looping coefficients," *Proceedings of the Royal Academy in Amsterdam*, vol. 15 (1913), pp. 113-122. Compare also: G. de Rham, "Sur l'analysis situs des variétés à n dimensions," *Journal de Mathématique*, vol. 10 (1931), pp. 115-200; in particular pp. 156-165; H. Seifert, "Verschlingungsinvarianten," *Sitzungsberichte der Preussischen Akademie, Physik.-Math. Klasse*, 1933, pp. 811-828.

invariant properties of M_n if the complementary dimensions k and $n-k-1$ are distinct, since in that case any base may be used for the torsion group (the group of zero-divisors) both for the dimension k and the dimension $n-k-1$.

If $n=2k+1$, however, the complementary dimensions are equal and the following problem arises. Let two manifolds M_n and M'_n have simply isomorphic groups of k-dimensional torsion. In which case is it possible to conclude from their looping coefficients alone that M_n and M'_n are not homeomorphic. This problem may be solved by transforming the matrix $L(a_i, a_j)$ into one of a certain collection of non-equivalent normal forms, by a suitable choice of the base a_i for the group of k-dimensional torsion. It is easily seen[2] that in considering this problem one may consider separately for each given prime number p, the zero-divisors of which the order is a power of p.

For even k, the problem was solved by G. de Rham.[3] It turns out that no new invariant of the manifold may be obtained in this case by the consideration of zero-divisors of which the order is a power of an odd prime number p, while for the zero-divisors of order 2^k certain unexpected a priori possibilities arise for the matrix $L(a_i, a_j)$. However thus far no examples are known of manifolds for which the matrix $L(a_i, a_j)$ assumes such an anomalous form.

For odd k and $p \neq 2$, the problem was solved by H. Seifert.[4] In this note the case of odd k will be considered without the restriction that $p \neq 2$. Certain of the considerations of Seifert will be repeated (cf. **2**-**5**). This is due partly to certain modifications necessitated by the inclusion of the case $p=2$ and partly to the desire to treat the cases $p \neq 2$ and $p=2$ in the same way. In contrast to the former case, the main difficulty of the latter case is the consideration of the influence on each other of zero-divisors of which the orders are adjoining powers of 2 (cf. **11**).

In **12** the normal forms arrived at are computed for an infinite collection of known 3-dimensional manifolds. These examples clearly indicate how little the homology and intersection theory can contribute to the fundamental problems concerning manifolds even of such a low dimension.

1. The looping coefficient $L(a,b)$ of a k-dimensional zero-divisor a of order α and a $(n-k-1)$-dimensional zero-divisor b of order β in an orientable manifold M_n is given by

$$L(a,b) = S(c,b)/\alpha, \tag{1}$$

[2] H. Seifert, *loc. cit.*, pp. 816-817.
[3] *Loc. cit.*, pp. 159-165.
[4] *Loc. cit.*, p. 823.

where c has αa as boundary. Here $S(c, b)$ is the intersection number and a and c do not have a point in common. The following properties are well known:

(2) $L(a, b)$ is uniquely determined modulo 1 if only the homology classes of a and b are given.

(3) $L(a, b) \equiv (-1)^{kn+1} L(b, a)$ modulo 1.

(4) $L(\alpha_1 a_1 + \alpha_2 a_2, b) = \alpha_1 L(a_1, b + \alpha_2 L(a_2, b)$.

(5) If $\alpha a \sim 0$ and $\beta b \sim 0$, then $L(a, b)$ may be written as a fraction with the greatest common divisor (α, β) of α and β as denominator. In particular if α and β are relative prime, then $L(a, b)$ is an integer.

(6) If a_i, $i = 1, 2, \cdots$ are the k-dimensional torsion numbers of M_n, then they are also the $(n-k-1)$-dimensional torsion numbers of M_n. Furthermore if a_i is a base of the group of k-dimensional zero divisors, such that $a_i a_i \sim 0$, then a base b_i may be found for the $(n-k-1)$-dimensional zero divisors in such a way that $\alpha_i b_i \sim 0$ and $\alpha_i L(a_i, b_j) = \delta_{ij}$. Here δ_{ij} is the Kronecker symbol.

(7) A statement analogous to (6) remains true for those k-dimensional zero-divisors of which the order is a power of a fixed prime number p.

This statement follows easily from the last remark in (5). The same remark shows also that the problem under consideration is solved in the general case if it has been solved for the zero-divisors of prime power order corresponding to every prime divisor of the torsion numbers of M_n.

2. From now on, k is an odd integer, $n = 2k + 1$ and M_n is a fixed orientable n-dimensional manifold. For a fixed prime number p, let G denote the group formed by these k-dimensional zero divisors of G, of which the order is a power of p. Let $a_1, \cdots, a_2$ be a base of G and let every relation between the a_i be a consequence of the order relations

(8) $$p^{e_i} a_i \sim 0, \quad 0 \leqq e_1 \leqq e_2 \leqq \cdots \leqq e_s.$$

Let A be the matrix with elements $g_{ij} = p^{e_i} L(a_i, a_j)$; then (2) shows that the integers g_{ij} are uniquely determined modulo p^{e_i}, while (3) shows that

(9) $$p^{e_j} g_{ij} \equiv p^{e_i} g_{ji} \text{ modulo } p^{e_i + e_j}.$$

Since the transformation from one base of G to another base of G may be made by means of a non-singular matrix modulo p, it is clear that (7) is equivalent to the statement that A is non-singular modulo p. Since, by (9),

g_{ij} is divisible by $p^{e_i - e_j}$, whenever $e_i > e_j$, the symmetric submatrices $A^{(e)}$ of A, which correspond to sets of subscripts $l, l+1, \cdots, m$, where

$$(10) \qquad e = e_l = e_{l+1} = \cdots = e_m, \quad e_{l-1} < e, \quad e_m < e_{m+1},$$

are also non-singular modulo p.[5] Now $A^{(e)}$, being non-singular modulo p, has an inverse matrix modulo p^e which also has integer elements. This inverse may be obtained by multiplying the ordinary inverse by an appropriate integer prime to p. If $e^i > e^m = e$ then the sum of a_i and any linear combination of the generators $a_l, a_{l+1}, \cdots, a_m$ may be used as generator instead of a_i. With the help of the inverse matrix of $A^{(e)}$ one may easily find linear combinations in such a way that for the new matrix A,

$$g_{ji} = g_{ij} = 0, \quad j = l, \quad l+1, \cdots, m, \quad i > m.$$

Thus A may be reduced to diagonal form, each element of the diagonal being one of the matrices $A^{(e)}$.

If this reduction has been completed, let H be the subgroup of G generated by the elements $a_l = b_1, a_{l+1} = b_2, \cdots, a_m = b_r$ of G, $r = m - l + 1$. Then any relation between the b_i is a consequence of the order relations $p^e b_i \sim 0$. Let $[c_i, c_j]$ denote, for any pair of elements c_i, c_j of H, the number $P^e L(c_i, c_j)$. Then, by (1), (2), and (4) the $[c_i, c_j]$ are integers uniquely determined modulo p^e and the $[c_i, c_j]$ are linear in c_i. Since k and n are odd, (3) implies that $[c_i, c_j] = [c_j, c_i]$.

Furthermore, the matrix

$$(11) \qquad B = \| h_{ij} \| = \| [b_i, b_j] \| = \| [b_j, b_i] \| = \| h_{ji} \|$$

is non-singular modulo p, since it is identical with $A^{(e)}$.

The next problem is to determine a number of non-equivalent normal forms into one of which every such matrix B may be transformed by a change of base of H. If the new generators of H are given in terms of the old generators by left multiplication with a matrix P, which is necessarily non-singular modulo p, then the matrix B is replaced by the matrix PBP'. Thus the determinant of B may be multiplied by any quadratic residue modulo p^e but not by any other factor.

In what follows a matrix B will occasionally be considered as belonging to two or more groups H of which the generators have two or more distinct orders p^e. Wherever necessary the exponent e will be specified by placing modulo p^e after a statement on B.

3. Let B be the matrix (11) corresponding to a group H which is generated by r independent elements $b_1, \cdots, b_r$, each of order p^e.

[5] H. Seifert, *loc. cit.*, pp. 818-819.

First it will be shown [6] that B *may be transformed into diagonal form if p is odd.* In fact, if h_i is not divisible by p, one has $[b_i, mb_i] = mh_{ii} \equiv 1$. modulo p^e for a suitably chosen integer m. If one replaces b_j by the new generator $b_j - mb_ih_{ij}$, the element h_{ij} will be replaced by zero. Proceeding in this fashion one may replace all elements in the i-th row and i-th column (except h_{ii} itself) by zero for any i, for which $h_{ii} \not\equiv 0$ modulo p. Now suppose that all reductions of this type have been made and that h_{ii} is divisible by p. Since B is non-singular modulo p, it follows that at least one element in the i-th row, say h_{ij}, is not divisible by p. Since all reductions of the preceding type have been made, h_{jj} must be divisible by p. If one replaces b_i by the new generator $b_i + b_j$, the i-th diagonal element in the new matrix B is $h_{ii} + h_{jj} + 2h_{ij}$. Thus if $p \neq 2$, the new diagonal element is not divisible by p and the preceding reduction may be continued until B has been transformed into diagonal form.

4. Suppose that $p \neq 2$ and that B has been transformed into diagonal form. If u_1 and u_2 are arbitrary diagonal elements of B, corresponding to the generators b_1 and b_2 of H, then it is possible to replace u_1 by 1, changing u_2 if necessary, but leaving all other elements of B unchanged.[7] It will first be shown that the subgroup of H generated by the elements b_1 and b_2 contains an element $c = \alpha_1 b_1 + \alpha_2 b_2$ such that

$$(12) \qquad [c, c] = \alpha_1{}^2 u_1 + \alpha_2{}^2 u_2 \equiv 1 \text{ modulo } p^e.$$

If u_1 (or u_2) is a quadratic residue of p^e, one may choose α_2 (or α_1) equal to zero and determine α_1 (or α_2) in such a way that (12) is satisfied. If u_1 and u_2 are not quadratic residues of p^e, let v be a quadratic non-residue of p^e such that for some r,

$$(13) \qquad 1 + r^2 \equiv v \text{ modulo } p^e.$$

One may take as v the smallest positive quadratic non-residue modulo p. For any quadratic residue (non-residue) modulo p is also a quadratic residue (non-residue) modulo p^e. The group of prime residual classes modulo p^e is cyclic, so that $u_1 v$ and $u_2 v$ are quadratic residues. Thus the integers a_1 and a_2 may be found in such a way that

$$\alpha_1{}^2 u_1 v \equiv 1, \qquad \alpha_2{}^2 u_2 v \equiv r^2 \text{ modulo } p^e.$$

Equation (12) is now an immediate consequence of (13).

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[6] H. Seifert, *loc. cit.*, p. 822.

[7] Compare H. Seifert, *loc. cit.*, p. 823, footnote. It seems that the elaboration in 4 is necessary to complete the indications given there.

The element c may obviously be used as one of the generators of the subgroup of H generated by b_1 and b_2. By (12) the corresponding diagonal element of the new matrix B is 1. If the other generator of the subgroup H is chosen according to the method of **3**, and the remaining generators of H are left unchanged, the object stated at the beginning of **4** is obviously reached.

Clearly one may use the above method to replace the diagonal elements of B, except perhaps one, by the number 1. The remaining element different from 1 may be placed in the position h_{11} and may be multiplied with an arbitrary factor α^2 prime to p^e by multiplying the corresponding generator with α. Thus it is clear that *a matrix B corresponding to a triple of numbers $p \neq 2$, e, r may be transformed into the unit matrix E_r, if* det B *is a quadratic residue modulo p^e, and B may be transformed into the matrix $E_r^{(p)}$, obtained from the unit matrix E_r by changing the first diagonal element into an arbitrary quadratic non-residue modulo p^e, if* det B *is not a quadratic residue modulo p^e. Moreover, if $p \neq 2$, then two matrices, which are non-singular modulo p, are equivalent modulo p^e, if and only if they are equivalent modulo p.*

5. The structure of the group H is determined by the numbers p, e, r, which in turn are determined by (10) in terms of the fundamental relations (8) of G. However, it is clear that the subgroup H of G is not only not uniquely determined by G but even not uniquely determined by the matrix A of the numbers g_{ij}. In fact let A be reduced, as described in **2**, to a diagonal form, each diagonal element of which is one of the matrices $A^{(e)}$. If i and j are such that $e_i > e_j$, add $p^{e_i - e_j}a_i$ to the generator a_j of G. Apply the reduction of **2** to the mattrix A in its new form. The j-th generator remains equal to $a_j + p^{e_i - e_j}a_i$, so that the group H, belonging to $e = e_j$, has been replaced by a different group. Accordingly the considerations of **3** and **4**, for the case $p \neq 2$, must be completed by an investigation of the changes of det $A^{(e)}$, when the group H is replaced by a different group. This is very easily done by means of the following statement.

If two different matrices B and C both are obtained from the same group G as the matrix $A^{(e)}$ corresponding to two different subgroups H and H', both of which have r generators of order p^e, then they satisfy a relation of the form

$$C \equiv PBP' \text{ modulo } p^c \tag{14}$$

for some matrix P which is non-singular modulo P. Here c is the least positive value of $| e_i - e |$ and the e_i are the different orders of generators of G.

In fact the generators b'_i of H' may be expressed in terms of the generators b_i of H, $i = 1, \cdots, r$ and the remaining elements a_j of a set of generators

of G (which includes the b_i) in the form $b' = Pb + Qa$, where b', b and a are vectors with components b'_i, b_i and a_j. It is easily verified that P must be a non-singular square matrix modulo p, so that new generators of H are defined by $b'' = Pb$. The matrix $A^{(e)}$ which corresponds to the new generators b'' is clearly $D = PBP'$.

Thus in order to prove (14) it is sufficient to show that if the group H is modified by the addition of linear combinations of generators of order $\neq p^e$ to the generators b'' of order p^e, then the congruence classes modulo p^e of the corresponding matrices $A^{(e)}$ are the same.

If a_j has the order $p^{e_j} < p^e$ and if the generator b''_i is replaced by $b'_i = b''_i + \alpha a_j$, then one has for the i-th diagonal element of $A^{(e)}$,

$$p^e L(b'_i, b'_i) = p^e L(b''_i, b''_i) + p^{e-e_j}[2\alpha p^{e_j} L(a_j, a_j) + \alpha^2 p^{e_j} L(a_j, a_j]$$

and similar changes take place in the other elements of B. If a_j has the order $p^{e_j} > p^e$, then only linear combinations of the form $\alpha p^{e_j - e} a_j$ may be added to the generators of H, and a similar change in the elements of $A^{(e)}$ results. Since passage from b''_i to b'_i is possible by a finite number of steps of the two types described it clearly follows that (14) holds.

As a direct consequence of (14) and the last statement in **4** one obtains the following final statement for the case under consideration: *If $p \neq 2$, the invariants of the different matrices $A^{(e)}$ are also invariants of A itself.*

6. Let B be the matrix (11) which corresponds to a group H with r generators each of order 2^e. *Then B may be transformed into diagonal form, if and only if at least one diagonal element of B is odd.* Since B is non-singular the necessity of this condition is made evident by the remark that, if every diagonal element of B is even, then $[a, a]$ is even for every a in H.

For the proof of the sufficiency let it be remarked first that, if no diagonal element of B is odd, then the number of rows of B is even. Otherwise B would be singular modulo 2, since every term of B would either contain a diagonal element or have a symmetric term distinct from itself. It also follows that the coefficient of any diagonal element of B in det B is even, so that B remains non-singular modulo 2^e if any number is added to one of its diagonal elements.

If B has at least one odd diagonal element, B may be reduced by the method of **4** to the form

$$B = \| h_{ij} \| = \begin{pmatrix} h_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & h_s & \\ & & & & B'' \end{pmatrix}$$

where B'' is a square matrix without odd diagonal element and the empty

spaces represent terms which are equal to zero. If one replaces the generator b_{s+1} by $b_{s+1}+b_1$, then $h_{1\,s+1}$, $h_{s+1\,1}$ and $h_{s+1,s+1}$ increase by h_1, so that the new $(s+1)$-th diagonal element is an odd number. Since the matrix, which replaces B'', is non-singular by the above remark, one can reduce the elements $h_{1\,s+1}$ and $h_{s+1\,1}$ to zero by adding a suitable linear combination of $b_{s+1},\cdots,b_r$ to b_1. The first diagonal element in the resulting matrix B is again odd, since it is the only element $\neq 0$ in its column. Thus the number of odd diagonal elements of B has been increased by one. This procedure, combined with the method in **3**, reduces B to diagonal form.

7. Let B be the matrix (11) belonging to a group H with r generators of order 2^e and suppose that B has diagonal form. If $e=1$, B has the form E_r modulo p^e, so that no further reduction is necessary. If $e=2$, the diagonal elements of B may be taken equal to $+1$ or -1. It will be shown that *a matrix B corresponding to a triple of numbers $p=2$, $e=2$, r may be transformed into one of the non-equivalent forms E_{rm}, $0\leqq m<3$, $m\leqq r$, where the first m elements in the diagonal of E_{rm} are -1, while the remaining elements are $+1$,* (in particular $E_{r0}=E_r$). First it is easily seen that any four elements equal to -1 in the diagonal of B may be replaced simultaneously by $+1$. In fact if $[a_i,a_i]=-1$, $i=1,2,3,4$, then $[b_i,b_i]\equiv+1$ modulo 4, where

$$b_1=a_1+a_2+a_3,\ b_2=a_1-a_2+a_4,\ b_3=-a_1+a_3+a_4,\ b_4=a_2-a_3+a_4.$$

Thus if the b_i replace the a_i as generators of H, four elements -1 in the diagonal of B are replaced by $+1$. This proves that any of the matrices B under consideration may be transformed into one of the above normal forms E_{rm} and also that *m is determined as the residual class modulo 4 of the number of diagonal elements -1 in B.* It remains to prove that the above normal forms E_{rm} are not equivalent. The value of $\det B$ may be used to prove that E_{rm} and E_{rn} are not equivalent if $m-n$ is odd. The non-equivalence of E_{q0} and E_{q2} and also of E_{q1} and E_{q3} is a consequence of the non-equivalence of E_{r0} and E_{r2} if $r>q$. This is seen by adding $r-q$ generators of order 2^e to the q generators which give rise to E_{qm}.

Thus it remains to prove the non-equivalence of E_{r0} and E_{r2} for some arbitrarily high r. For any group H of the type under consideration, let H' be the subgroup of H determined by the elements of order 2 of H. The value of $[a,a]$ is constant on each co-set of H' in H. In the table below the number of these co-sets, on which $[a,a]$ takes specified values, is listed for five groups E_{rm}. If the matrix B, belonging to H, is E_r, the number of co-sets of H' in H, in which $[a,a]\equiv\alpha$ modulo 4, is obviously $\sum_{0\leqq 4l+\alpha\leqq r}\binom{r}{4l+\alpha}$. This gives the first, third and fourth columns of the table. The second may be easily

verified directly. The last may be obtained from the second and third by noting that the group H corresponding to $E_{r+1\,2}$ is the direct sum of the groups corresponding to E_{r0} and E_{22}. The table shows immediately that the matrices $B = E_{r0}$ and $B = E_{r2}$ are not equivalent for any r of the form $8k + 2$.

$[a, a] \diagdown B$	E_2	E_{22}	$E_{8k\,0}$	$E_{8k+2\,0}$	$E_{8k+2\,2}$
0	1	1	2^{8k-2}	$2^{8k} - 2^{4k}$	$2^{8k} + 2^{4k}$
1	2	0	$2^{8k-2} + 2^{4k-1}$	2^{8k}	2^{8k}
2	1	1	2^{8k-2}	$2^{8k} + 2^{4k}$	$2^{8k} - 2^{4k}$
3	0	2	$2^{8k-2} - 2^{4k-1}$	2^{8k}	2^{8k}

8. Let B be the matrix (11) of a group H with r generators of order $2^e > 4$, and suppose that B may be transformed into diagonal form. *Then B is equivalent to the matrix $B' = \| h_i \delta_{ij} \|$ modulo 2^e if and only if B is equivalent to B' modulo* 8. For $e = 3$ this is evident. Suppose that the statement has been proved for $e - 1$, so that B may be transformed modulo 2^{e-1} into the form B'. The same transformation transforms B modulo 2^e into the form $B' + 2^{e-1}D$, where each element of D is 0 or 1. If B is reduced to diagonal form by the method in **3**, it retains the form $B' + 2^{e-1}D$, where D is now a diagonal matrix each element of which is 0 or 1. The proof by induction of the above statement in italics is completed by the remark that $1 + 2^{e-1}$ is a quadratic residue modulo 2^e for $e > 3$, since

$$5^{2^c} \equiv 1 + 2^{c+2} \text{ modulo } 2^{c+3}, \qquad c = e - 3 \geqq 0.$$

Thus it remains to consider the diagonal matrix B if $2^e = 8$. In that case *any two diagonal elements h_1 and h_2 of B may be replaced simultaneously by $h_1 + 4$ and $h_2 + 4$.* In fact it is sufficient to replace the corresponding generators b_1 and b_2 of H by $b_1 + 2h_1b_2$ and $b_2 - 2h_2b_1$. Thus, if B is equivalent modulo 4 to the form E_{rm}, it is equivalent modulo 8 (and also modulo 2^e, $e > 3$) either to E_{rm} or to F_{rm}, where F_{rm} is obtained from E_{rm} by adding 4 to the first diagonal element. That E_{rm} and F_{rm} are not equivalent is evident, since their determinants are distinct modulo 8. Accordingly, *one has for the matrix B corresponding to a group H with r generators of order $2^e > 4$ the eight (or $2r + 2$ if $r \leqq 3$) non-equivalent diagonal normal forms E_{rm}* and *F_{rm}, $0 \leqq m \leqq 3$, $m \leqq r$.*

The following statement may be obtained easily from the above considerations: *Two matrices B and B', which belong to two groups with r generators of order $p^e = 2^e \geqq 8$ and have at least one odd diagonal element, are equivalent modulo 2^e if and only if they are equivalent modulo* 8.

9. Let B be the matrix (11) belonging to a group H with r generators b_i of order 2^e and suppose that every diagonal element of B is even, so that B cannot be transformed into diagonal form and r is even. If h_{ij} is an odd

element of B, the submatrix of B formed by the elements in the intersections of the i-th and j-th column with the i-th and j-th row is non-singular modulo 2, so that the remaining elements in these rows and columns may be changed into zeros by adding multiples of b_i and b_j to the other generators of H. Continuing this process one may transform B into a diagonal matrix the elements of which are two rowed square matrices. The odd elements in these two-rowed matrices may be made equal to 1 by multiplying one of the two corresponding generators with some odd number. Then each such matrix appears in the form

$$C=\begin{pmatrix}2k & 1\\ 1 & 2l\end{pmatrix}. \tag{15}$$

If two matrices of the form (15) *are congruent modulo* 4, *then they are equivalent modulo* 2^e. If $e=1$ any two matrices of the form (15) are equivalent. If $e=2$ their equivalence modulo 2^e is an immediate consequence of their congruence modulo $4=2^e$. Thus the above statement will be proved by complete induction if it is shown that, if two matrices (15) are equivalent modulo 2^{e-1}, they are equivalent modulo 2^e, $e>2$. If C is the one matrix, and the other may be transformed into C modulo 2^{e-1}, then it may be transformed into the form $C+2^{e-1}D$ modulo 2^e, where D is symmetric and each of its elements is either 0 or 1. Now any diagonal element of D may be made equal to 0, since, if $e>2$, one may add 2^{e-1} to a diagonal element of $C+2^{e-1}D$ by adding to the corresponding generator 2^{e-2} times the other generator. Multiplying, if necessary, one of the generators by $1+2^{e-1}$ one finds that C and $C+2^{e-1}D$ are equivalent, so that the above statement in italics follows by complete induction.

Any matrix (15) *is equivalent modulo* 4, *and hence modulo any power of* 2, *to one of the two matrices*

$$I_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad J_2=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}. \tag{16}$$

In fact, since equivalence modulo 4 is a consequence of congruence modulo 4 and generators may be interchanged, it is sufficient to show that the matrices I_2 and

$$\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}$$

are equivalent modulo 4. This is evident if one replaces the second generator by the sum of the two generators.

The four rowed matrices

$$\begin{pmatrix}I_2 & 0\\ 0 & I_2\end{pmatrix} \text{ and } \begin{pmatrix}J_2 & 0\\ 0 & J_2\end{pmatrix}$$

are equivalent modulo 2^e. For $e = 1$ this is evident. If $e > 1$ transform the second matrix by adding the third generator of the corresponding group to the first generator. The upper left-hand corner then takes a form equivalent to I_2 and may be reduced to I_2. If the matrix is reduced further to normal form, the lower right-hand corner must assume either the form I_2 or the form J_2. That it cannot assume the form J_2 follows from the fact that I_4 and J_4 are not equivalent. This will be proved in **10**, where I_r and J_r are defined for every even r.

10. If r is even, let I_r denote the matrix obtained by arranging $\frac{1}{2}r$ matrices I_2 in the diagonal of an r-rowed matrix, of which the remaining elements are zero, and let J_r denote the matrix obtained from I_r by replacing the first of the matrices I_2 by J_2. It is easily seen that the considerations in **9** imply:

If B is the matrix (11) *of a group H with r generators of order 2^e and if every diagonal element of B is even, then r is even and B may be transformed into the form I_r or J_r.* It will next be proved that *I_r and J_r are equivalent modulo 2^e if and only if $e = 1$.* The case $e = 1$ of this statement is evident. It is also evident that the case $e > 2$ is a consequence of the case $e = 2$. It remains to show that I_r and J_r are not equivalent modulo 4. But this is evident from $\det I_r \not\equiv \det J_r$ modulo 8 and from the following statement.

If B has even diagonal elements and B is replaced by an equivalent matrix modulo $2^e = 4$, then the residual class of $\det B$ *modulo* 8 *remains unchanged.* In fact, changing B into an equivalent matrix either results in multiplying $\det B$ by the square of an odd constant (and such a square is always congruent to 1 modulo 8), or it consists in adding multiples of 4 to the elements of B. The latter changes do not affect the residual class of $\det B$ modulo 8, since B has to remain symmetric, has even diagonal elements and has an even number of rows.

The results obtained may be restated as follows: *Two matrices B and B', which belong to two groups with r generators each of order $p^e = 2^e \geqq 4$, and one of which has no odd diagonal element, are equivalent modulo 2^e if and only if they are equivalent modulo* 4.

If a matrix B belongs to a group H, which has r generators of order 2^e, $e > 1$, and if every diagonal element of B is even, then B is equivalent to I_r or J_r according as $\det B \equiv (-1)^{\frac{1}{2}r}$ *or* $\det B \equiv 5(-1)^{\frac{1}{2}r}$ *modulo* 8.

11. Let G be a group of which all elements have an order which is a power of 2 and let the corresponding matrix A (cf. **2**) be transformed into normal form by the methods of **6-10**. As in **5**, it is necessary to investigate in how far this normal form is dependent on the particular choice of the sub-

groups H of G corresponding to the different orders 2^e of base elements of G. By (14) and the statements at the end of **8** and **10** the invariant properties of a matrix $A^{(e)}$ derived from A, are also invariant properties of A if $e = 1$, or if $e = 2$, $c > 1$ or if $c > 2$ (Immediately below (14) the definition of c may be found).

The problem described above will be investigated here in the case where only two distinct exponents e occur in the orders 2^e of base elements of G and if the difference of these two exponents is not more than 2, i. e. if in (14), $c \leqq 2$. On the basis of the facts obtained, it will be easy to decide what are the invariant properties of any particular matrix A.

The following three statements will be needed:

(17) *Modification modulo* 4 *of a matrix B in the normal form I_r or J_r or modification modulo* 2 *of non-diagonal elements of such a matrix does not change the normal form.* The first part of this statement has already been proved. The second part depends of course on the fact that all matrices under consideration are symmetric. It is easily checked that the addition of one pair of symmetrically placed even elements to I_r or J_r does not change the residual class of the determinant modulo 8. But the same must hold if more than one such pair is added. For any new term, which contains one element each of more than one pair of new elements, cannot affect the determinant modulo 8, since it is either divisible by 4 and duplicated by a symmetric term, or else is symmetrically placed, in which case it is divisible by 16.

(18) *Modification modulo* 8 *of a matrix B in the normal form E_{rm} or F_{rm} or modification modulo* 4 *of non-diagonal elements of such a matrix does not affect the normal form.* The first part has already been proved. For the second part it should be remembered that m is invariant under modifications modulo 4. The statement follows, since it is easy to verify that the determinant modulo 8 is not changed by addition of pairs of symmetrically placed numbers which are divisible by 4.

(19) *Modification modulo* 4 *of a matrix B in the normal form E_{rm} or F_{rm} or modification modulo* 2 *of non-diagonal elements of such a matrix may interchange E and F but cannot change the value of m.* The first part is already known. For the second part, notice that if a pair of symmetrically placed even elements is added to E_{rm} or F_{rm} and this pair is eliminated by a change of base according to **3**, then the change in diagonal elements is a modification modulo 4, so that the second part is a consequence of the first part.

In what follows a matrix A is supposed to be built up out of a matrix $A^{(e)}$ and a matrix $A^{(e+1)}$ or $A^{(e+2)}$, both of which are supposed to have normal form and the problem is to determine the invariant properties of A. The

large number of cases to be considered has been arranged in five groups of which only the last gives some trouble.

The form $A^{(e)}A^{(e+2)}$ *cannot be modified if at least one of the normal forms is* I_r *or* J_r. That the one equal to I_r or J_r cannot be modified follows from (14) and (17). But if one of the matrices has the form E_{rm} or F_{rm}, its normal form cannot be modified by (18), which is applicable, since the other matrix has even diagonal elements.

The form $A^{(e)}A^{(e+1)}$ *cannot be modified if both matrices have a normal form of the type* I_r *or* J_r. In fact in this case the second half of (17) is applicable, since all diagonal elements are even.

In the form $A^{(e)}A^{(e+2)}$, *where both* $A^{(e)}$ *and* $A^{(e+2)}$ *have normal forms of the type* E_{rm} *or* F_{rm}, *the two values of* m *and the product of the determinants of* $A^{(e)}$ *and* $A^{(e+2)}$ *modulo* 8 *are invariants.* In other words both $A^{(e)}$ and $A^{(e+2)}$ may change from E to F or conversely at the same time but not separately. That the two values of m are invariant follows from (14) and (19). That the product of the determinants is invariant modulo 8 is evident, since the transformation of generators for $H^{(e)}$ and $H^{(e+2)}$ combined consists in multiplication with a non-singular matrix. It remains to show that the determinant of $A^{(e+2)}$ is not invariant. This is clear, since the determinant of $A^{(e+2)}$ changes if any generator b_i of $H^{(e+2)}$ is replaced by the sum of b_i and any generator of $H^{(e)}$.

In the form $A^{(e)}A^{(e+1)}$, *where one matrix is of the type* I_r *or* J_r *and the other is of the type* E_{sm} *or* F_{sm}, *the value of* m *and the product of the two determinants modulo* 8 *are the only invariants.* In other words the one may change from I_r to J_r or conversely if the other changes at the same time from E_{sm} to F_{sm} or conversely. That m is invariant follows from (19). The product of the determinants is always invariant modulo 8. In order to see that the determinants of $A^{(e)}$ and $A^{(e+1)}$ are not separately invariant in case $A^{(e+1)}$ has the form I_r or J_r, it is sufficient to add one generator of $A^{(e)}$ to both generators corresponding to a matrix I_2 or J_2 in $A^{(e+1)}$. If $A^{(e)}$ has the type I_r or J_r, one obtains the same result by adding twice one generator of $A^{(e+1)}$ to a similar pair of generators of $A^{(e)}$.

If $e = 1$ or if $e = 2$ and $A^{(e)}$ is of type E or F in the last two statements, the product of the determinants modulo 8 is not an invariant of the matrix A, since then $\det A^{(e)}$ is not an invariant of $A^{(e)}$.

Let the two matrices $A^{(e)}$ $A^{(e+1)}$ *both have a normal form of type* E *or* F. In (20) each vertical pair of numbers represents diagonal elements the first in $A^{(e)}$, the second in $A^{(e+1)}$. Two vertical pairs separated by the sign $\sim$ are such that the one pair may be replaced by the second in the normal form under discussion. If $e = 1$ or $e = 2$ the table may be simplified in an obvious way.

$$(20)\qquad \begin{array}{lcccccccc} e: & 1\;3 & 1\;7 & 1\;3 & 1\;7 & 3\;5 & 3\;5 & 5\;7 & 5\;7 \\ & \sim & \sim & \sim & \sim & \sim & \sim & \sim & \sim \\ e+1: & 1\;3 & 3\;5 & 5\;7 & 7\;1 & 1\;7 & 5\;3 & 1\;3 & 5\;7 \end{array}$$

In order to see how (20) is obtained, consider the generators a and b of order 2^e and 2^{e+1} with $L(a,a)=k2^{-e}$, $L(a,b)=0$, $L(b,b)=l2^{-e-1}$. The new generators $a'=a+2b$, $b'=kb-la$ satisfy $L(a',a')=(k+2l)2^{-e}$, $L(a',b')=0$, $L(b',b')=(k^2l+2l^2k)2^{-e-1}$. Since in every $A^{(e)}$ any reduction modulo 8 may be obtained by a change of generators and since k and l are odd numbers, one sees that the pair k, l may be replaced by the pair $k+2l$, $l+2k$. That no other equivalences are valid, except those in (20) may be verified easily. Now if $A^{(e)}A^{e+1}$ is a matrix A, and if both $A^{(e)}$ and $A^{(e+1)}$ are of the form E_{rm} or F_{rm}, then it is easily seen from (20) that $A^{(e+1)}$ may be reduced to the form E_{r0} or F_{r0} and also that, if $A^{(e)}$ and $A^{(e+1)}$ do not both have exactly one row, *then* $A^{(e+1)}$ *may be reduced to the form* $E_{r0}=E_r$. *The invariants of the new form of* $A^{(e)}$ *are now the invariants of the pair* $A^{(e)}\,A^{(e+1)}$. This statement may be proved by a method which is closely related to the one used at the end of **7**. Of course if both $A^{(e)}$ and $A^{(e+1)}$ have one row and $e>1$, then the pair $A^{(e)}\,A^{(e+1)}$ has in addition the invariant property that $A^{(e+1)}$ may be reduced to the form E_{10} or F_{10}.

12. Examples may be given to show that all normal forms found above for the matrix of looping coefficients of a manifold of dimension $n=4k+3$ with given $2k+1$ dimensional torsion may actually occur. The problem to prove this is simplified very much by the well known remark that the sum of two manifolds M_n and M'_n has a matrix of looping coefficients which appears in the form

$$\begin{pmatrix} L & 0 \\ 0 & L' \end{pmatrix}$$

where L and L' are the matrices of looping coefficients of M_n and M'_n. The sum of two manifolds of the same dimension n is obtained by omitting an n-simplex from both manifolds and identifying the boundaries of the remaining bounded manifolds. Thus any normal form of a matrix of looping coefficients which is diagonal may be obtained by forming the sum of suitably chosen lens spaces [8] (Linsenräume) and all other normal forms may be obtained if examples are known of manifolds with 2 torsion numbers 2^e, such that their matrices of looping coefficients have the normal forms I_2 and J_2.

In case $n=3$ many examples which are not sums of simpler manifolds may be found among Seifert's "gefaserte Räume." Here only the manifolds of that type will be considered with a signature [9] of the form

[8] Compare H. Seifert, *loc. cit.*, pp. 824-825.

[9] For the definition and properties of "gefaserte Räume," see H. Seifert, *Acta*

(21) $$(O\,0;0\,|\,0;\alpha_0,2^e;\alpha_1,2^e;\cdots,\alpha_r,2^e),$$
where
(22) $$r \text{ is even, } \alpha_i \text{ is odd, } i=0,1,2,\cdots,r,\ \alpha_0=1.$$
Manifolds with such signatures are obtained by the following construction.

Let $C\times S$ be the product of a simple closed curve C and a 2-dimensional manifold S of genus zero. Let C_i, $i=0,\cdots,r$, be $r+1$ simple closed curves on S of which the "interiors" have no point in common. Now omit for each i the product of C and the interiors of C_i and replace this product by a different anchor ring T_i with the same boundaries $C\times C_i$ but on which $2^e b_i+\alpha_i a$ is homologous to zero instead of b_i. Here a is a basic cycle on $C\times p$ (p is any point of S) and b_i is a basic cycle on $q\times C_i$ (q is any point of C).

Let u_i be a 2-chain which is on T_i, except for the product of S, and an arc from a point of C_i to the point p, and suppose that
(23) $$u^i \rightarrow 2^e b_i+\alpha_i a.$$
Then one has for the intersection number $S(u_i,b_j)$ of u_i and b_j,
(24) $$S(u_i,b_j)=\alpha_i\delta_{ij},$$
where b_j is thought of as moved into the interior of T_j by a small deformation. As generators of the homology group of the manifold under consideration we may use b_i and a, with the basic homologies
(25) $$2^e b_i+\alpha_i a\sim 0,\quad i=0,\cdots,r,\quad\text{and}\quad \sum_{i=0}^{r} b_i\sim 0.$$
The last relation expresses the fact that the "exterior" of all curves C_i has as its boundary the sum of all simple closed curves C_i. As new generators of the same homology group one may use
(26) $$a \text{ and } c_j,\ \text{where } c_j=\alpha_j b_0-b_j,\ j=1,\cdots,r.$$
For (25) and (26) imply
$$\sum_{j=1}^{r} c_j\sim\beta b_0\ \text{ and }\ 2^e c_j\sim 0,\ j=1,\cdots,r,\ \beta=\sum_{i=0}^{r}\alpha_i.$$
By (22), the coefficient β of b_0 is an odd number. Thus b_0 may be obtained as a linear combination of $\beta b_0\sim\sum_{j=1}^{r} c_j$ and $2^e b_0\sim -a$. Since $b_j=\alpha_j b_0-c_j$, this implies that every element of the homology group may be obtained as a linear combination of the elements (26). The basic homology relations (25), if expressed in terms of the generators (26), take the form
(27) $$\beta a\sim 0,\ 2^e c_j\sim 0,\ j=1,\cdots,r.$$

Mathematica, vol. 60 (1933), pp. 147-238. For the signature (21), see in particular p. 181. For the statement that the spaces are not sums of simpler manifolds, see p. 229.

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The α_i are not required to be positive. Accordingly one may choose the α_i in such a way that $\beta = \sum_{i=0}^{r} \alpha_i = 1$, in which case the manifold has r torsion numbers equal to 2^e. Whether or not one makes this assumption, the only group H of interest has the r generators c_i of order 2^e. The corresponding matrix B may be determined as follows. By (23) and (26)

$$\alpha_i u_0 - u_i \rightarrow 2^e(\alpha_i b_0 - b_i) = 2^e c_i, \quad i = 1, \cdots, r.$$

Hence on denoting again by h_{ij}, $i, j = 1, \cdots, r$ the general element of B,

$$h_{ij} = S(\alpha_i u_0 - u_i, \alpha_j b_0 - b_j),$$

and so, by (24),

$$(28) \qquad B = \| h_{ij} \| = \| \alpha_i \alpha_j + \alpha_i \delta_{ij} \|, \quad i, j = 1, \cdots, r.$$

Since all diagonal elements of B are even by (22), the normal form of B is I_r or J_r according as

$$(29) \qquad \det B \equiv (-1)^{\frac{1}{2}r} \text{ or } \det B \equiv 5(-1)^{\frac{1}{2}r} \text{ modulo } 8,$$

and before determining $\det B$ one may reduce the elements of B modulo 4. So suppose that

$$(30) \quad \alpha_i \equiv 1 \text{ modulo } 4,\ i \leqq s; \quad \alpha_i \equiv -1 \text{ modulo } 4,\ i > s; \quad r - s = t.$$

Then B reduces modulo 4 to the form

$$\begin{pmatrix} A_{ij} & C_{lj} \\ C_{ik} & D_{lk} \end{pmatrix} \quad i, j = 1, \cdots, s; \ l, k = s+1, \cdots, s+t,$$

where

$$A_{ij} = 1 + \delta_{ij}, \quad C_{lj} = C_{ik} = -1, \quad D_{lk} = 1 - \delta_{lk}.$$

In order to compute the determinant, first change the sign of the last t columns, then subtract the $(i+1)$-th column from the i-th column successively for $i = 1, \cdots, s+t-1$. Next subtract suitable multiples of the first up to the $(s+t-1)$-th column from the last column, making all elements of the last column zero except the last element. Thus one sees that

$$(31) \qquad \det B \equiv (-1)^t (s - t + 1) \text{ modulo } 8.$$

Comparing (29) and (31) the following statement may be verified:

If a manifold with signature (21) *satisfies* (22) *and* (30), *then its matrix of looping coefficients has the normal form I_r if $s - t \equiv 0$ or 6 modulo 8 and J_r if $s - t \equiv 2$ or 4 modulo 8.*

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FIXED POINTS AND THE EXTENSION OF THE HOMEOMORPHISMS OF A PLANAR GRAPH.*

By Saunders MacLane and Virgil W. Adkisson.

1. **Introduction.** Gehman, Adkisson, and Moore [1] have found conditions under which a given homeomorphism of a continuous curve on the sphere can be extended to the sphere. In this paper we consider the problem [2] of characterizing those graphs on the sphere for which every homeomorphism of the graph can be extended to the sphere. We restrict ourselves to cyclicly connected [3] graphs and obtain a necessary and sufficient condition that such a graph has some one map [4] on the sphere in which all of its homeomorphisms are extendable. This necessary and sufficient condition depends essentially on the behavior of certain fixed points, for it asserts merely that the fixed points of all homeomorphisms of order 2 must be distributed on the graph as are the fixed points of periodic homeomorphisms of the sphere.

1.1. Let G_1 be a cyclicly connected continuous curve lying on the sphere and consisting of a finite number of simple closed curves. Then G_1 is composed of a number of simple arcs $\alpha_1, \beta_1, \cdots$, each ending on two branch points. These simple arcs and branch points, considered as elements, constitute a *combinatorial graph* G; such a graph is made up of a finite number of *edges* (simple arcs) $\alpha, \beta, \gamma, \cdots$ and a finite number of *vertices* $p, q, r, \cdots$, each edge "ending" on just two vertices.[5] An edge α with ends p and q will be denoted by $\alpha(p, q)$. The original curve G_1 is a *map* of the graph G; for a given G_1, we may assume, without loss of generality, that each vertex of G is on at least three edges of G.

A given homeomorphism T of G_1 to itself takes each arc α_1 of G_1 into some arc $\beta_1 = T\alpha_1$ and thus sets up a one-to-one correspondence between the

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[1] Gehman, [1], [2], [3]; Adkisson, [1], [2]; Moore, [1]. Numbers in brackets refer to the bibliography at the end of the paper.

[2] One form of this problem was originally suggested by J. R. Kline.

[3] For this concept see Whyburn, [1], [2], [3], and Whyburn and Kuratowski [1]. A cyclicly connected graph is also non-separable in the sense of Whitney. See Whitney, [1], Theorem 7.

[4] In another paper we characterize graphs such that all homeomorphisms are extendable in every map. See Adkisson and MacLane [1].

[5] For definitions of other combinatorial terms used, see Whitney [1] or [2], or MacLane [1].

corresponding edges α and β in the combinatorial graph G. This correspondence is an automorphism of G. Here an *automorphism* σ of a combinatorial graph G is a one-to-one correspondence $\alpha \leftrightarrow \sigma\alpha$ of the edges of G to the edges of G and a one-to-one correspondence $p \leftrightarrow \sigma p$ of the vertices p of G to the vertices of G such that each edge $\alpha(p, q)$ corresponds to an edge $\sigma\alpha$ with the ends σp and σq.

The extendability of a homeomorphism is intimately related to the possibility of disconnecting the graph by removing two of its vertices. We shall call this "splitting" the graph. More explicitly, a *semi-split* of G at the vertices h_1 and h_2 is a representation of G as a sum

$$G = H + H', \tag{1}$$

where H and H' are two non-void subgraphs, having in common no arcs and no vertices except for the vertices h_1 and h_2. This semi-split is a *split* of G if neither H nor H' is a chain. A subgraph H in a semi-split (1) of G at h_1 and h_2 will be called *least* if H has itself no semi-split at h_1 and h_2; that is, if any two arcs of H are connected outside h_1 and h_2. A split (1) of G will itself be called *least* if either of the components H or H' of the split is least. If (1) is a least split, the subgraphs H_1 and H_2 are *components* of G, and each component has the *ends* h_1 and h_2. Note that component means "component of a least split." [6]

1.2. If a graph G is mapped on the sphere, and if a homeomorphism of the sphere carries G into itself, then a fixed point x of the homeomorphism on the sphere may be on the graph, in which case the graph will have a fixed vertex or a fixed edge. Otherwise x lies in one of the complementary domains into which the graph subdivides the sphere, so that the boundary of this complementary domain must go into itself in the homeomorphism. If G is cyclicly connected, this boundary is a circuit of G. Hence fixed points on the sphere appear in G either as fixed vertices, edges, or circuits. We define for any automorphism σ of G: a *fixed vertex* p is one with $\sigma p = p$. A *fixed edge* α is one with $\sigma\alpha = \alpha$, while a circuit C is a fixed circuit if $\sigma C = C$, and if C contains no fixed vertices or edges. We note that in a fixed circuit C the cyclic order of the vertices and edges must be preserved by σ, since an inversion of this order would necessarily involve two fixed points, either edges or vertices, on C. A *combinatorial fixed point* will mean either a fixed vertex, edge, or circuit.

[6] It can be shown that a cyclicly connected graph G has a split if and only if it contains a "split circuit," in the sense of this term used in Adkisson and MacLane, [1].

1.3. A homeomorphism of a graph which can be extended to the sphere gives essentially [7] a periodic transformation T of the sphere into itself. But a periodic transformation is homeomorphic to a rotation about an axis or to a reflection in a diametral plane or to a product of a rotation with a reflection, and hence has either two fixed points, or a circuit all of whose points are fixed, or else no fixed points (cf. Eilenberg, [1]). Some impossible configurations would thus be

(i) a circuit of fixed points plus another fixed point;

(ii) three distinct fixed points plus a fixed circuit.[8]

Our main theorem states essentially that a graph with no such impossible configurations has a map in which all homeomorphisms are extendable. In the census of distinct fixed points for condition (ii), several fixed circuits might unfortunately correspond to the same fixed point on the sphere. In this case the fixed circuits will "nest" in some fashion. More explicitly, three fixed circuits, each pair of which lies on one side of the third, would correspond to three distinct fixed points on the sphere. In this statement, the term "side" refers to the map of the graph on the sphere. We can replace it by a purely combinatorial term. A *combinatorial side* of a circuit C in G is a subgraph $H \subset G$ such that G has a least split $G = H + (G - H)$ in which [9] $C \subset (G - H)$.

We thus obtain the following condition which refers only to combinatorial properties of the graph. In stating this condition an exception is made of the particular graph G_0, consisting of two vertices p and q, and of four distinct arcs, each with the ends p and q.

THEOREM 1. *If G is a cyclicly connected planar graph, not the graph G_0 defined above, and if each vertex of G is on at least three edges, then G can be so mapped on the sphere that any homeomorphism of G into itself can be extended to a homeomorphism of the sphere into itself if and only if there is no automorphism σ of G with $\sigma^2 = 1$ and with either one of the following invariant configurations in G:*

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[7] More explicitly, T^n for some n will carry every complementary domain of the graph into itself.

[8] The presence of a fixed circuit excludes the possibility of representing σ by a reflection of the sphere. For, if σ is of order 2, the assertion that there is a fixed circuit is equivalent to the assertion that there is a chain A containing neither fixed vertices nor fixed edges and having the ends p and σp. This means that there is some vertex p not separated from its image vertex σp by the set S of all fixed edges and vertices, while in a reflection in a diametral plane the fixed edges and vertices do separate every vertex in one hemisphere from its image in the other hemisphere.

[9] $G - H$ denotes the subgraph of G containing all edges of G not in H.

(i) *A circuit C, all of whose vertices and edges are invariant, and a fixed edge or vertex not on C;*

(ii) *A fixed circuit C and three distinct combinatorial fixed points, x, y, and z, such that if any one of x, y, or z is a circuit*[10] *D, then the other two of x, y, and z are both contained in one combinatorial side of this circuit D.*

1.4. Since the condition of Theorem 1 refers only to automorphisms of order 2, it will have the following curious consequence.

THEOREM 2. *If a planar combinatorial graph G is cyclicly connected and has each vertex on at least three edges, and if G has no automorphisms of order 2, then G has a map in which every homeomorphism is extendable to the sphere.*

It is more remarkable that the condition of Theorem 1 refers only to properties of individual homeomorphisms and not to the relation between several homeomorphisms. This can be stated as follows:

THEOREM 3. *If the planar combinatorial graph G is cyclicly connected, has each edge on three or more vertices, and if G has no map on the sphere in which every homeomorphism is extendable, then there is one homeomorphism T of G which can not be extended in any map of G.*

This theorem is true even for the graph G_0, for here the automorphism cyclicly permuting three edges and leaving the fourth edge fixed is not extendable in any map.[11]

1.5. The proof of Theorem 1 will first be reduced to a purely combinatorial problem. This will be done in § 2, where we also treat a simpler but analogous problem which shows the essential rôle of the "splits." The treatment of the combinatorial problem will then proceed by mathematical induction. The essential device consists in obtaining from a given graph G a simpler graph, the "skeleton" of G, which has all of the homeomorphisms of G. The existence and properties of this skeleton will be discussed in §§ 4 and 5, while the subsequent paragraphs will show by a suitable subdivision into cases that the map of the skeleton can generate a corresponding map of

[10] If x is a circuit, it is a fixed circuit, so that the presence of the fixed circuit C need not be explicitly required.

[11] For any $G \neq G_0$, the homeomorphism T of Theorem 3 can be taken to be of order 2, but for G_0 there is no such "nowhere extendable" automorphism of order 2. This unique behavior of the graph G_0 justifies the apparently capricious exceptional rôle of this graph in our main theorem.

the original graph in which the homeomorphisms will be extendable. This proof of the sufficiency of the condition of Theorem 1 is supplemented in § 3 by a simple topological proof for the necessity of the condition.

2. Splits versus extendability. In this paragraph we consider the problem of the extendability of a homeomorphism which carries a graph into some other graph, with the following conclusion:

THEOREM 4. *If G_1 and G_2 are two maps of a cyclicly connected combinatorial graph G on the spheres S_1 and S_2 respectively, then every homeomorphism T of G_1 to G_2 can be extended to a homeomorphism of the sphere S_1 to S_2 if and only if G has no split.*

2. 1. A given homeomorphism T establishes an automorphism of G, provided each vertex of G is on at least three edges. For if α_1 is any arc of G_1, and if α_1 and $T\alpha_1$ are maps of the edges α' and α'' respectively of G, then the correspondence $\sigma\alpha' = \alpha''$ yields an automorphism of G.

The extendability of a homeomorphism T to the sphere depends upon combinatorial properties of the corresponding automorphism. The boundaries of the complementary domains into which the cyclicly connected graph G_1 divides the sphere (call them *c. d. boundaries*) are the maps of circuits of G.[12] These circuits control extendability as follows:

THEOREM 5. *A homeomorphism T of a cyclicly connected graph G_1 on a sphere S_1 to a graph G_2 on a sphere S_2 can be extended to a homeomorphism of S_1 to S_2 if and only if T carries every c. d. boundary of G_1 to a c. d. boundary of G_2.*

This theorem is an easy extension of Theorem 2 of Adkisson, [1], which was proven from a theorem of Gehman, [1]. In this case, since the number of domains is finite, we can also proceed thus: Let R_1 and R_2 be regions of S_1 and S_2 respectively whose boundaries C_1 and C_2 correspond under σ_1. Use the result that the interior and boundary of a Jordan curve can be mapped on the interior and boundary of a circle to extend σ_1 so that it carries R_1 into R_2. The combination of these extensions is a homeomorphism of S_1 to S_2.

2. 2. If a graph G has no splits the c. d. boundaries of its maps on the sphere are uniquely determined. For these boundaries are simply[13] the

[12] See Whyburn, [1], Theorem 10 or MacLane, [2], Theorem 5. 3. The latter theorem gives a complete combinatorial characterization of these circuits.

[13] G is either triply connected or a θ-graph. For a θ-graph, the statement above about c. d. boundaries is immediate. For a triply connected graph it is given in MacLane, [3], Theorem 6.

circuits C in G whose removal from G does not disconnect G. Hence any automorphism must carry these circuits into themselves, so that any homeomorphism of G_1 necessarily preserves the c. d. boundaries and is therefore extendable by Theorem 5. This proves the sufficiency of the condition of Theorem 4.

2. 3. For the necessity proof we need the fact that in a map of a split graph any component can be "turned around" on the two split vertices. If G is split at p and q into components H_1 and H_2, we introduce two new arcs $\alpha_1(p,q)$ and $\alpha_2(p,q)$, and form $H_1+\alpha_1$, $H_2+\alpha_2$. In any map of $H_1+\alpha_1$ there must be two c. d. boundaries containing α_1, so that these boundaries may be written in the form $L_1+\alpha_1$ and $M_1+\alpha_1$, where L_1 and M_1 are two chains in H_1 with ends p and q.

LEMMA 2. 4. *If $G=H_1+H_2$ is a split at p and q, and if $H_1+\alpha_1(p,q)$ has a map with the c. d. boundaries*

$$C_1, C_2, \cdots, C_s, \quad L_1+\alpha_1, M_1+\alpha_1, \quad (L_1\subset H_1, M_1\subset H_1) \tag{2}$$

while $H_2+\alpha_2(p,q)$ has a map on the sphere with the c. d. boundaries

$$D_1, D_2, \cdots, D_t, \quad L_2+\alpha_2, M_2+\alpha_2, \quad (L_2\subset H_2, M_2\subset H_2), \tag{3}$$

then there is a map of G in which the c. d. boundaries are the circuits

$$C_1, C_2, \cdots, C_s, \quad D_1, D_2, \cdots, D_t, \quad L_1+L_2, M_1+M_2, \tag{4}$$

and another map of G with the c. d. boundaries

$$C_1, C_2, \cdots, C_s, \quad D_1, D_2, \cdots, D_t, \quad L_1+M_2, M_1+L_2. \tag{5}$$

Proof. In the map of H_2 on the sphere, remove α_2 and insert in its place [14] the given map of H_1. This gives one of the maps (4) and (5). The same process, with a change in notation, gives the second map.

The two maps (4) and (5) are distinct, for were these sets of c. d. boundaries identical, we should have either $M_1=L_1$ or $M_2=L_2$. But $M_1=L_1$ means that the chain $M_1=L_1$ is the whole of H_1, contrary to the definition of a split. Therefore, the homeomorphism of G which carries the first map of any edge into the second map of the same edge does not preserve c. d. boundaries, and so is not extendable. Hence all homeomorphisms are extendable only if G has no split, as asserted in Theorem 4.

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[14] For details, cf. MacLane, [3], Lemma 5.

3. The necessity proof. The proof that the combinatorial conditions of our main theorem are necessary requires first a demonstration of the relevance of the term "combinatorial side" used in our theorem. We shall show that a combinatorial side always lies within a topological side.

3.1. LEMMA. *If H is a combinatorial side of a circuit C in G, then in any map of G on the sphere all of H, except perhaps for two of its vertices, lies on one side of the map of C.*

Proof. By definition of a combinatorial side H, G has a least split $G = H + (G - H)$ at two vertices h_1 and h_2, while $C \subset (G - H)$. Therefore C can meet H at most in the ends h_1 and h_2 of H. Suppose that H is not all on one side of C.

In the given split, either H or $G - H$ is least. If H is least, and if E_1 is the part of H on one side of C, then $H = E_1 + (H - E_1)$ is a semi-split of H_1 because these two non-void parts can meet only in the vertices h_1 and h_2. This contradicts the assumption that H is least.

Suppose then that $G - H$ is least. If α and β are edges of H on opposite sides of C, then, because G is cyclicly connected, there is a circuit D in G containing α and β. This D must cross C to get from α to β. But α and β are in H, so that D, which has vertices on C and hence in $G - H$, must pass through the split vertices h_1 and h_2. Therefore D crosses C at h_1 and h_2 and lies entirely in H. Because $G - H$ is least, $G - H$ must as in the previous case lie entirely on one side of D. Hence C, which is in $G - H$, also lies on one side of D, contrary to the fact that D crosses C. This contradiction shows that H must lie on one side of C and can meet C at most in its end vertices h_1 and h_2. The lemma is established.

3.2. We now show that the conditions (i) and (ii) of Theorem 1 are necessary if the graph G has a map in which every homeomorphism is extendable. For suppose that G has such a map in which each edge α has a map $M\alpha$ and also that G has an automorphism σ of order 2. For this σ one can readily construct a corresponding homeomorphism T which will also have period 2 and which will have the same fixed points as σ. More explicitly, each fixed vertex of σ is fixed under T, each fixed edge α of σ is point-wise fixed or contains a fixed point under T according as σ leaves fixed or interchanges the ends of α, and each fixed circuit C maps into a simple closed curve MC such that $T(MC) = MC$ and such that T has no fixed points on MC. By assumption the transformation T can be extended to the sphere. Therefore, as in Theorem 5, T carries every c. d. boundary of the map M into

a c. d. boundary of this map, while T certainly has period 2 when considered as a permutation of these boundaries. T can then be readily extended to a homeomorphism T^* of the whole sphere in such a way that T^* has the same period 2.

Suppose now that σ has the impossible configuration (i) of Theorem 1: A circuit C of edges and vertices all of which are fixed and a fixed edge or vertex not on C. Corresponding to these, T will have a simple closed curve $M(C)$ consisting of fixed points and also a fixed point not on C. But this is impossible because T^* is a periodic homeomorphism of the sphere, for the Fundamental Lemma of Eilenberg, [1], states that a periodic transformation of the surface of the sphere has either (1°) no fixed points, (2°) all fixed points, (3°) a simple closed curve of fixed points and no other fixed points, (4°) exactly two fixed points.

In the second place consider an automorph σ with the fixed circuit C and the combinatorial fixed points x, y, and z of condition (ii) of the theorem. If x, y, and z are all fixed edges or vertices, then T will have three distinct fixed points. By the result of Eilenberg quoted above this is possible only if T^* has a simple closed curve of fixed points. In this case (Eilenberg, [1], Theorem 1), T^* is homeomorphic to a reflection of the sphere in a diametral plane. But the map of the given fixed circuit C cannot cross this diametral plane, for C contains neither fixed edges nor vertices. Thus $M(C)$ lies on one side of the diametral plane, contrary to the fact that $\sigma C = C$.

In the remaining cases one or more of the given fixed points x, y, and z may be fixed circuits. If x is a circuit D, then y and z lie on a combinatorial side of D. Lemma 3.1 then shows that $M(y)$ and $M(z)$ lie on one side of $M(D)$. Call the side of $M(D)$ not containing $M(y)$ and $M(z)$ the "inside" of $M(D)$. Since σ leaves fixed y and z "outside" D, the extended T^* must carry the inside of $M(D)$ into itself. But a periodic transformation of a circle and its interior is homeomorphic either to a rotation or to a symmetry in a diameter (Eilenberg, [1], Theorem 2). Because D has neither fixed edges nor vertices, T^* is not a symmetry in a diameter of D, and so must be homeomorphic to a rotation of the inside of $M(D)$, so that T^* has a fixed point "inside" $x = D$. This fixed point is distinct from y and z. If y or z is a circuit, T^* likewise has a fixed point within this circuit, so that in any event T^* has three distinct fixed points. This gives a contradiction, as in the previous case, and this contradiction establishes the necessity of the conditions of Theorem 1.

This necessity could have been established without appeal to the general theorem of Eilenberg, the proof of which involves complexities of a point-set-

theoretic character. In the present case the given periodic transformation T^* of the sphere is associated with a "triangulation" G, such that T^* is combinatorially periodic on this triangulation. This fact can be used to give for our theorem a proof depending on a readily established combinatorial substitute [15] for Eilenberg's theorem.

4. The construction of the skeleton. The sufficiency proof of our theorem proceeds by an induction, reducing a given graph to a simpler graph having essentially the same group of automorphisms. This simpler graph, called a "skeleton," will be obtained by replacing a "minimal" component of the original graph by a single edge and by simultaneously replacing all homologous minimal components by new edges.

4.1. A component M of a least split of G is *minimal* if and only if in every least split $G = H + H'$ of G the component M is contained entirely in H or entirely in H'. We now deduce several requisite properties of minimal components. The definition immediately proves the property

4.2. *Two minimal components are either identical or else have at most end vertices in common.*

4.21. *In any least split $G = E_1 + E_2$ at the vertices e_1 and e_2 the component E_1 is either cyclicly connected or else E_1 contains another component of G with fewer edges than E_1.* For, if E_1 is not cyclicly connected, then E_1 has a "cut vertex" s such that there is a "separation" $E_1 = F_1 + F_2$, in which the non-void subgraphs F_1 and F_2 have only the cut vertex s in common. We shall show that one of the F's is a component of G. F_1 can touch the rest of G at most in the vertices s, e_1, and e_2. If F_1 contains only one of these three vertices, it is cut from $G - F_1$ by this vertex, contrary to the cyclic connectivity of G. If F_1 contains all three of these vertices, only one of them, s, can be in F_2, and this is similarly impossible. Therefore s, e_1, and e_2, are distinct, and F_1 touches the rest of G at just two vertices s and e_1. Hence $G = E_2 + F_1 + F_2$, where the only common vertices of these subgraphs are e_1 and e_2 in E_2, s and e_1 in F_1, s and e_2 in F_2. This gives two semi-splits

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[15] This substitute runs as follows: Let G be a cyclicly connected graph mapped on the sphere such that (i) each vertex of G is on at least three edges; (ii) σ is a homeomorphism of the sphere to itself with $\sigma G = G$; (iii) σ^2 but not σ carries every edge and vertex of G into itself; (iv) C is a c. d. boundary of the map of G, $\sigma C = C$, and σ preserves cyclic order on C. Then there is in addition to C either an edge or a vertex or a c. d. boundary of G which is carried into itself by σ, and there is only one such invariant element (vertex, edge, or boundary). Furthermore, if D is a c. d. boundary with $\sigma D = D$, then σ preserves cyclic order on D.

(1) $$G = (E_2 + F_1) + F_2, \qquad G = (E_2 + F_2) + F_1.$$

If neither were a split, both F_1 and F_2 would be chains, so that E_1 would be a chain, contrary to assumption. Both $E_2 + F_1$ and $E_2 + F_2$ are least because each has a cut vertex (at e_1 or e_2 respectively). Therefore one of the decompositions (1) is a least split, so that either F_1 or F_2 is a component of G contained in E_1, as asserted.

4.22. *Every minimal component of a cyclicly connected graph is cyclicly connected.* For a component not cyclicly connected contains by the above argument a proper subcomponent of G and hence cannot be minimal according to our definition.

4.23. *Every component E_1 of a least split $G = E_1 + E_2$ contains a minimal component of G.* We shall show that E_1 is either minimal or contains a component of G with fewer edges than E_1. From this the assertion will follow by induction. We denote the ends of E_1 by e_1 and e_2. By § 4.21, we may assume that E_1 is cyclicly connected. Furthermore, we may assume E_1 least, for if E_1 is not least, then either E_1 contains a smaller least component with the same ends e_1 and e_2, as required for our induction, or else E_1 consists of $t \geqq 2$ suspended chains with ends e_1 and e_2, while E_2 is least. In this case, the given split $G = E_1 + E_2$ is the only least split of G at e_1 and e_2, while in any other least split of G at vertices p and q, neither p nor q could lie inside any suspended chain of E_1, so that p and q do not disconnect E_1. In any least split of G, E_1 is, therefore, contained in one component, so that E_1 is in fact minimal.

In the remaining cases we have an E_1 which is least and cyclicly connected. If E_1 is not minimal, there is a split $G = H_1 + H_2$ at h_1 and h_2 such that E_1 is in neither H_1 nor H_2. Since arcs of E_1 are disconnected by h_1 and h_2, and since E_1 is cyclicly connected, both of h_1 and h_2 must be in E_1. As E_1 is least, one of h_1 and h_2 is not an end of E_1. Either E_2 contains neither h_1 nor h_2, or else E_2 contains one of h_1 or h_2 at one end of E_2. But all edges of E_2 are connected to both ends e_1 and e_2, so that E_2 will in no event be disconnected by h_1 and h_2. Therefore E_2 is all contained in one component of the split $G = H_1 + H_2$, say in H_1. Hence $H_2 \subset E_1$. Because H_2 and E_1 have distinct ends, H_2 has fewer edges than E_1, so that we have the component H_2 contained in E_1, as required.

4.24. Let E be a component from a least split of G at e_1 and e_2. Form the graph $E + \alpha$ where $\alpha(e_1, e_2)$ is a new edge not in G. *Then E is minimal if and only if* (i) *each end of E is on at least two edges of E*; (ii) $E + \alpha$ *has*

no least split. This fact will show that the decision whether a given component E is minimal or not depends only on the structure of $E + \alpha$ and not on the rest of G. This graph $E + \alpha$ can be considered as a "homomorphic" image of G obtained by replacing all of $G - E$ by a single edge α with the same ends. It is then geometrically apparent and it can be readily verified combinatorially that any least split of $E + \alpha$ will yield a corresponding least split of G, if we replace α by $G - E$. Conversely, any least split of G at two vertices p and q will yield a "homomorphic" least semi-split of $E + \alpha$, provided both p and q are in E, and provided E is not all contained in one component of the given least split.

By this means we prove the necessity of the conditions above: (ii) is requisite, because any least split of $E + \alpha$ would give a homomorphic split of G in which E did not all lie in one component, so that E would not be minimal. (i) is necessary because E, to be minimal, must by § 4. 22 be cyclicly connected. Conversely, suppose (i) and (ii) to hold. Then E must be cyclicly connected, for were E to have a separation at a cut vertex s, then $E + \alpha$ must have a corresponding least semi-split which would be a split because of (i). This split would contradict (ii). Next, if E were not minimal there would be a least split $G = H + H'$ at vertices p and q, such that both H and H' contain edges of E. Since E is cyclicly connected, both the vertices p and q of this split must be in E. This split therefore gives a homomorphic least split of $E + \alpha$ at p and q, contrary to condition (ii). Therefore conditions (i) and (ii) are sufficient to make E minimal.

4. 3. Two types of minimal components may be found by classifying graphs $M + \alpha$ with no least splits as "branch graphs" and "triply connected graphs." A graph consisting only of $k \geqq 2$ suspended chains, $A_1, A_2, \cdots, A_k$ having the same two ends p and q, but not having any other vertices in common, will be called a *branch graph*. A cyclicly connected graph G may be called *triply connected* if it is not a branch graph and if it has no split. The definitions readily show that a graph with a split at two vertices must have a least split at the same vertices unless both original components were branch graphs. We deduce thence the following dichotomy:

4. 31. *A cyclicly connected graph which has no least split is either triply connected or is a branch graph.* This fact, combined with § 4. 24, then yields

4. 32. *If M is a least and minimal component, then $M + \alpha$ is triply connected, while if the minimal component M is not least, then M is a branch graph with ends m_1 and m_2, and $G - M$ is least.*

4.4. Consider now a cyclicly connected graph G with a group Σ of automorphisms and with some least split. Because of the existence of minimal components (§ 4.23), we can then find a minimal component $M = M_0$ in G. Any automorphism of the group must carry M_0 into another subgraph having the same combinatorial properties, that is, into another minimal component of G. Let

$$(2) \qquad M_0, M_1, M_2, \cdots, M_{k-1}, \qquad (M_i = \sigma M_0, \sigma \epsilon \Sigma)$$

be all distinct minimal components which arise in this fashion from M_0 by the automorphisms of Σ. These components, by § 4.2, have at most end vertices in common. Hence we can construct a new graph G' by replacing each M_i of (2) by a new arc μ_i with the same ends as M_i. The resulting graph G' we call a *skeleton* of G.

Since least splits of G never take place inside the minimal components M_i replaced in the formation of G', it is clear that least splits of G yield corresponding semi-splits of G', and conversely, as follows:

4.41. *If $G = H_1 + H_2$ is a least split of G at p and q, then G' has at p and q a least semi-split $G' = H'_1 + H'_2$, where each H'_j arises from H_j $(j = 1, 2)$ by replacing each M_i in H_j by the corresponding new arc μ_i.*

4.42. *If $G' = F'_1 + F'_2$ is a split of the skeleton G' at p and q, then G has at p and q a split $G = F_1 + F_2$, where each subgraph F_j is obtained from F'_j $(j = 1, 2)$ by replacing each new arc μ_i in F'_j by the corresponding M_i. If the given split of G' is least, then the resulting split of G is least.*

4.43. *A skeleton of a cyclicly connected graph is cyclicly connected.* This fact is readily established from the definitions.

4.5. Corresponding to the given group Σ of automorphs of G, we now define a group Σ' of automorphs of the skeleton G'. Each $\sigma \epsilon \Sigma$ induces a corresponding transformation σ' of G'. For, each edge μ_i of G' corresponds to a component M_i of G, and σM_i must be some one of the set (2) of homologous minimal components. If $\sigma M_i = M_j$, we set $\sigma' \mu_i = \mu_j$ for the corresponding new edges μ_i and μ_j, while if α is an edge of G' which is also in G we set $\sigma' \alpha = \sigma \alpha$. These equations, with $\sigma' p = \sigma p$ for any vertex p of G', define the induced transformation σ' on G'. This σ' is a $1 - 1$ transformation carrying vertices and edges of G' into vertices and edges of G'. It is an automorphism, because σ' is so constructed that an edge $\alpha(p, q)$ has an image $\sigma' \alpha$ with ends $\sigma' p$ and $\sigma' q$. Finally, the correspondence $\sigma \rightarrow \sigma'$ preserves the product of automorphs, for if $\rho = \sigma \tau$ holds in Σ, the definition shows that $\rho' = \sigma' \tau'$. Since

the product is preserved the set Σ' of all these σ' is a group and the correspondence $\sigma \rightarrow \sigma'$ is a homomorphism of Σ to Σ'. We shall subsequently show that this homomorphism is in certain cases an isomorphism.

4. 6. The skeleton which we have thus defined can be used in particular cases to explicitly construct all homeomorphisms of a graph, for we need only find the group of all homeomorphisms of the skeleton (which may be constructed by replacing simultaneously all minimal components) and investigate which of these can, in fact, be extended to be homeomorphisms of the graph. By taking repeated skeletons, the problem is thus reduced to that of finding homeomorphisms of a triply connected graph.

5. Admissible groups of automorphisms. The sufficiency proof for our main Theorem 1 proceeds by induction, using the skeleton, and requires conditions on the automorphisms of the skeleton which are in stronger form than the conditions in the theorem itself. A group of automorphisms satisfying these conditions (cf. I and II below) will be called "admissible," and we shall prove in the subsequent paragraphs that any G can be so mapped that all the automorphs of a given admissible group of automorphs can be extended to the sphere.

5. 1. To define an admissible group, we consider automorphisms which "twist" a single component or which permute several components "in parallel." An automorphism σ is said to *twist* G if there is a least split of G with a component H such that $\sigma H = H$ and σ carries each end of H into itself, while σ is not the identity on H.

A representation of G with more than two subgraphs

$$(1) \qquad G = H_1 + \cdots + H_m \qquad (H_i \neq 0, i = 1, \cdots, m)$$

is called a *multiple split* of G at the points p and q if any two distinct subgraphs H_i and H_j have in common no edges and only the vertices p and q, and if no H_i has a semi-split at p and q. An automorphism σ is a *parallel permutation* if there is a multiple split (1) at p and q such that $\sigma p = p$, $\sigma q = q$, $\sigma H_1 = H_2$ and such that either $m \geqq 4$ or else $m = 3$ and H_3 is not a chain.

A group Σ of automorphisms of a cyclicly connected graph G is *admissible* if and only if

(I) Σ contains no parallel permutation and no twist of order 2.

(II) Σ contains no σ of order 2 for which G contains a fixed circuit and

three distinct combinatorial fixed points, x, y, and z, where x is a circuit only if y and z are in one combinatorial side of x, with similar conditions when y or z is a circuit.

5. 2. LEMMA. *If G satisfies the conditions of Theorem* 1, *then the group Σ of all automorphisms of G is admissible.*

The proof uses primarily the fact that Σ contains all the automorphs. The condition (ii) of the Theorem is the same as (II) of the definition. Hence we need only consider the condition (I). Suppose first, contrary to (I), that Σ were to contain a σ of order 2, twisting a component H. The transformation τ which is equal to σ on H and which is the identity on $G-H$ must then be an automorph of G. This τ is of order 2. Since $G-H$ is not a chain it contains a circuit whose vertices and edges are all invariant under τ. If $G-H$ contains edges not in this circuit these edges are also fixed under τ and τ violates the condition (i) of Theorem 1. Therefore $G-H$ must be just a circuit, and this circuit must pass through both ends p and q of H, so that $G-$H consists merely of two suspended chains L and N with ends p and q. Since each vertex of G is on at least three edges, L is a single arc, as is N. Define a new automorphism ρ which is the identity on H and which interchanges L and N. As before, H contains a circuit of points fixed under ρ. Hence, by the argument used for τ, H must consist only of this circuit. $G=H+L+N$ consists of four arcs with the same end points p and q, so that G is the particular graph G_0 excluded in Theorem 1. This contradiction shows that G can have no twists of order 2.

Suppose on the other hand that G has a parallel permutation σ, defined as above. Introduce a new automorphism τ which is equal to σ on H_1 and to σ^{-1} on H_2, and which is the identity elsewhere. This τ is of order 2 and may be treated exactly as the τ in the preceding paragraph. This treatment establishes the lemma.

5. 3. LEMMA. *If Σ is an admissible group of automorphisms for the cyclicly connected graph G and if G' is a skeleton of G with a corresponding group Σ' of automorphisms, then the correspondence $\sigma \to \sigma'$ defined in* § 4. 5 *makes Σ isomorphic to Σ'.*

The correspondence will be an isomorphism if no $\sigma \neq 1$ corresponds to the identical automorphism of Σ'. Suppose contrarywise that $\sigma \neq 1$ corresponds to $\sigma' = 1$. This means that $\sigma M_i = M_i$ for each minimal component M_i used in § 4, (2), to construct the skeleton, while σ is the identity except on these M_i. Consider the two types of minimal components established in § 4. 32.

In case M_i is a branch graph, then each one of the M's must be a branch graph. Select some one, say M_1, on which σ is not the identity and let M_1 consist of $k \geqq 2$ suspended chains $A_1, \cdots, A_k$ with ends p and q. Then σ must permute at least two of these A's, so that σ is a parallel permutation, contrary to the definition of an admissible group.

There remains the case when each one of the components M_i is least. Consider one such component M_1 with ends p and q, and adjoin a new arc $\alpha(p, q)$. Then by § 4.32, $M_1 + \alpha$ is triply connected, so that all maps of $M_1 + \alpha$ have the same c. d. boundaries by § 2.2. There will be two such boundaries $A + \alpha$ and $B + \alpha$ containing α. If we extend σ by setting $\sigma\alpha = \alpha$, then σ is an automorphism of $M_1 + \alpha$ which carries the c. d. boundaries into c. d. boundaries and therefore takes the unordered pair $(A + \alpha, B + \alpha)$ into itself. One of the following alternatives holds:

$$\sigma A = A,\ \sigma B = B; \quad \text{or} \quad \sigma A = B,\ \sigma B = A.$$

Under the first alternative, σ must leave every edge and vertex of these circuits $A + \alpha$ and $B + \alpha$ invariant. But σ can be extended from $M_1 + \alpha$ to a periodic homeomorphism of a sphere into itself. Because σ has more than a circuit of fixed points, it must by Eilenberg's results be the identity on $M_1 + \alpha$. In the second alternative, $\sigma^2 A = A$, $\sigma^2 B = B$, so σ^2 is, as in the previous alternative, the identity on M_1. σ^2 is thus the identity on every M_i, and so σ must actually be of order 2. But we have just shown that σ twists at least one M_i, which contradicts the condition (I) of an admissible group. Hence $\sigma \neq 1$ implies $\sigma' \neq 1$, and Σ' is isomorphic to Σ.

5.4. LEMMA. *If Σ is an admissible group for the cyclicly connected graph G, and if G' is a skeleton of G, then the corresponding group Σ' is an admissible group on the skeleton G'.*

To show that Σ' satisfies condition (I), suppose first that Σ' contains a σ' of order 2 which twists a component H'. Then the corresponding σ is by the preceding lemma also of order 2, while the component H of the original G corresponding to H' as in § 4.42 is itself twisted by σ, contrary to the assumption that the group Σ is admissible.

Secondly, suppose that Σ' contains a parallel permutation σ' for the multiple split

$$G' = H'_1 + \cdots + H'_m.$$

Then the argument of § 4.42 yields a corresponding decomposition

$$G = H_1 + \cdots + H_m$$

which is a multiple split of the original G. Since $\sigma' H'_1 = H'_2$, it follows that $\sigma H_1 = H_2$ for the corresponding automorph σ, so that σ is a parallel permutation, contrary to the assumption on Σ. This contradiction shows that the group Σ' satisfies the condition (I) for an admissible group. For the condition (II) we must consider how the ends of a minimal component can be interchanged.

5.41. LEMMA. *If M is a minimal component of G with ends m_1 and m_2, and if σ is an automorphism of order 2 with $\sigma M = M$, $\sigma m_1 = m_2$, then M contains either a fixed circuit under σ or else two fixed points under σ which are either vertices or edges.*

Consider the two possible types for M indicated in § 4.32. In case M is a branch graph, let A_1 denote one of the suspended chains of M. Then either $\sigma A_1 = A_1$, in which case A_1 contains either a fixed edge or vertex, or else $\sigma A_1 \neq A_1$, in which case $A_1 + \sigma A_1$ is itself a fixed circuit.

In case M is least then, as in the proof of the Lemma 5.3, any map of $M + \alpha$ contains two c. d. boundaries $A + \alpha$ and $B + \alpha$ which can at most be interchanged by σ. If $\sigma A = A$, then the hypothesis that the ends m_1 and m_2 of A are interchanged implies that A contains a fixed edge or vertex, while $\sigma B = B$, so that B contains a second fixed point. Otherwise $\sigma A = B$, and $A + B$ is a circuit [16] which is a fixed circuit, so that in all cases M contains a fixed point.

5.42. Suppose now that condition (II) fails for the group Σ'. Then Σ' contains an automorph σ' of order 2 with a fixed circuit C' and three combinatorial fixed points x', y', and z', restricted as in (II). We will obtain a contradiction by constructing corresponding fixed points in G.

In the given C' let p be any vertex and $\sigma' p$ its image, and let A' be a chain of C' with ends p and $\sigma' p$. Replace every "new" edge μ_i in A' by any chain N_i with the same ends as μ_i and taken from the corresponding minimal component M_i. These replacements in A' yield a chain A in G with ends p and σp. $A + \sigma A$ is readily seen to be a circuit C which is fixed under σ.

If the given fixed point x' is a circuit, the same treatment will yield a corresponding fixed circuit x in G. If x' is a vertex, the same vertex $x = x'$ is fixed in G. If x' is an edge α, then either α is also an edge of G and so a fixed point x of G, or else α is one of the new edges μ_i representing a com-

[16] For if $A + B$ is not a circuit A and B have in common a vertex s, $s \neq m_1$, $s \neq m_2$. An inspection of the map of $M + \alpha$, in which $A + \alpha$ and $B + \alpha$ are c. d. boundaries, shows that s must be a cut vertex of M, contrary to the cyclic connectivity of M (see § 4.22).

ponent M_i of G. Since $\sigma'\alpha = \alpha$, we then have $\sigma M_i = M_i$. If σ leaves fixed each end of M_i, these ends are fixed points in G. Otherwise, σ interchanges the ends, so that by § 5. 41 M_i does contain a fixed point x.

The given fixed points x', y', and z' of G' thus yield three distinct fixed points x, y, and z under σ in G. Furthermore, if x is a circuit, then y and z are on a combinatorial side of x. For the construction above gives a fixed circuit x only when x' is a circuit or when x' is an edge μ_i. If x' is a circuit, then, by the assumption that (II) is violated in G', y' and z' lie on a combinatorial side H' of x'. If all the edges μ_i in this component H' are replaced by the corresponding subgraphs M_i, we get as in § 4. 42 a corresponding "side" of x which contains the constructed fixed points y and z, as desired. In the other case, when x' is an edge μ_i, the new fixed circuit x lies in the corresponding minimal component M_i, and $G - M_i$ is a combinatorial side of x containing y and z.

For the automorph σ of order 2 in Σ we thus have a fixed circuit C and three distinct fixed points x, y, and z, contradicting condition (II) for Σ. This demonstrates the admissibility of Σ' asserted in Lemma 5. 4.

5. 5. To show the extent of the conditions for admissibility we note that they imply that an admissible group Σ contains no twist of any order, although the definition itself explicitly excludes only twists of order 2. In fact, the isomorphism of the groups Σ' and Σ shows that a twist of a given order must remain a twist of the same order on the skeleton,[17] so that any twist must eventually reduce to a twist of a minimal component in some skeleton, while such twists are either of order 2 or are parallel permutations and so are excluded.

6. Ambiguous automorphs. Certain automorphs θ in an admissible group may have so few fixed points that they can be realized geometrically either as rotations or as reflections. Such automorphs will be called "ambiguous."

6. 1. *Definition.* An automorph θ of order 2 in an admissible group Σ of G is ambiguous on the edge α of G if $\theta(\alpha) = \alpha$, if θ has no fixed circuits in G and if θ has in G exactly two combinatorial fixed points.

The fixed points of a given ambiguous θ will now be shown to divide G into two equal "halves," E_p and E_q. First, both ends of α can not be fixed points, for that would yield three fixed points. Hence θ turns $\alpha = \alpha(p, q)$

[17] Provided we use a skeleton with respect to the subgroup of Σ composed of the powers of the given twist.

end for end and has exactly one fixed edge or vertex x in addition to α. Let E_p be the subgraph containing all those edges β of G for which there exists a chain A_β free of fixed points, containing β, and ending on p. Define E_q similarly. Here a chain A_β is "free of fixed points" if and only if A_β contains no fixed edges and no fixed vertices, except perhaps for a fixed vertex at one end. Since $\theta p = q$ and since θ carries chains free of fixed points into chains free of fixed points, we have

$$(1) \qquad \theta E_p = E_q, \qquad \theta E_q = E_p.$$

6.2. *If G has an ambiguous θ with two fixed points $\alpha = \alpha(p, q)$ and x, then G has a semi-split*

$$(2) \qquad G = (E_p + \alpha) + (E_q + x).$$

Proof. First, every edge γ of G appears once on the right of (2). For, G is cyclicly connected, so that γ lies on a circuit C containing α. C can contain at most two fixed points α and x. If γ is neither of these, then the subchain of C joining γ to an end of α and not containing x will be free of fixed points. Therefore such a γ lies in either E_p or E_q.

E_p and E_q have no common edges. For suppose β were in both E_p and E_q. Then, by (1), $\theta\beta$ would be in both, and there would be chains A_β and $A_{\theta\beta}$ connecting β and $\theta\beta$ to p. $A_\beta + A_{\theta\beta}$ would necessarily contain a chain L free of fixed points and having ends r and s on β and $\theta\beta$ such that $\theta(r) = s$. If L contains an interior vertex t with $\theta t \epsilon L$, then replace L by the subchain joining t to θt. Eventually we would thus obtain a new chain L with no such interior vertices t and with ends r' and s' such that $\theta(r') = s'$. This means that $L + \theta L$ is a fixed circuit under θ, contrary to the Definition 6.1 which asserts that an ambiguous θ has no such fixed points.

We now know that each edge of G appears exactly once on the right of (2). But by definition of E_p two edges on a common vertex not a fixed point would both lie in the same component E_p or E_q. Therefore the only vertices common to E_p and E_q are fixed vertices. If the given fixed point x is a vertex, then the components $E_p + \alpha$ and E_q of (2) obviously have in common only the vertices x and q, so that (2) is a semi-split at x and q. If x is an edge $x = x(s, t)$, it cannot have both its ends in one of the subgraphs E_p or E_q, say in E_q, for then the two subgraphs of (2) would have but one vertex in common, so that this vertex would cut G, contrary to the cyclic connectivity of G. Hence an end s of x must lie in E_p while the other end is in E_q. In this case (2) is a semi-split at q and s.

The component $E_p + \alpha$ is least, for it contains only one edge α ending on the split vertex q. Therefore (2) is a least split of G unless E_p or E_q is a chain. But if E_p is a chain then $\theta E_p = E_q$ must also be a chain, so that G is simply a circuit. The following conclusions result:

6. 3. *If G has an ambiguous θ, then G has a least split or is a circuit.*

6. 4. *If θ is ambiguous on the edge α, then α is contained in no minimal component of G.* For a minimal component M would, by definition, be contained in one of the components $E_p + \alpha$, $E_q + x$ of (2). By symmetry, M is also contained in one of $E_q + \alpha$, $E_p + x$. Therefore M is in one of E_p, E_q, and hence outside α, as asserted.

6. 5. For a given map of G, we must distinguish when a given ambiguous automorph is a rotation and when it is a reflection. Hence the following definitions: Any automorph σ is *extendable* in a given map of G if each c. d. boundary C of the map is carried into a circuit, σC, which is a c. d. boundary. This means that any topological homeomorphism which corresponds to the automorphism σ as in § 1. 1 is extendable to the sphere (cf. Theorem 5 in § 2). Suppose a map is given in which every automorph of a group Σ is extendable. If θ is ambiguous on the edge α, then α is on two c. d. boundaries J and K, while $\theta(\alpha) = \alpha$. Because θ is extendable, θ carries J into one of J or K, and K into one of J or K. Define $\chi(\theta)$ as

$$\chi(\theta) = +1 \quad \text{if} \quad \theta(J) = K,\ \theta(K) = J, \tag{3}$$
$$\chi(\theta) = -1 \quad \text{if} \quad \theta(J) = J,\ \theta(K) = K.$$

If θ is also ambiguous on an edge β, α and β must by (2) belong to both J and K, and $\chi(\theta)$ is the same for α as for β. We call $\chi(\theta)$ the *character* of θ in the given map.[18] One verifies readily the following natural consequence of the definition.

6. 6. *If G has a map in which every automorph of a group Σ is extendable, and if $\theta \epsilon \Sigma$ is ambiguous on G, then any conjugate $\bar{\theta}$ of θ in Σ is also ambiguous on G, and $\chi(\bar{\theta}) = \chi(\theta)$.*

7. The regular mapping method. We now prove the following theorem by induction on the number of distinct least splits of G.

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[18] If $\chi(\theta) = +1$, then θ, when extended to the sphere, is homeomorphic to a rotation of the sphere, while $\chi(\theta) = -1$ would mean that θ yields a "reflection" of the sphere. Hence χ is in fact the group character of Σ which is $+1$ for all σ's which preserve orientation on the sphere and -1 for all other σ's.

THEOREM A. *If G is a cyclicly connected graph, and if Σ is an admissible group of automorphisms of G, then G can be mapped on the sphere so that every automorphism of Σ is extendable* (cf. § 6. 5). *Furthermore, if $\theta_1, \theta_2, \cdots, \theta_g$ are ambiguous automorphs of Σ on G, no two of which are conjugate in Σ, and if, for $i = 1, \cdots, g$, each ϵ_i is a number ± 1, then a map in which every $\sigma \epsilon \Sigma$ is extendable can be so chosen that each $\chi(\theta_i) = \epsilon_i$, for $i = 1, \cdots, g$.*

This asserts that the characters of the ambiguous automorphs can be arbitrarily assigned, subject to the inevitable restriction of § 6. 6.

7. 1. *First case of the proof:* G has no least split. This means as in § 4. 31 that G is either triply connected or a branch graph. If G is triply connected, the result of § 2. 2 shows that every homeomorphism is extendable in any map. If G is a branch graph, it consists of k suspended chains, $A_1, A_2, \cdots, A_k$, with the same ends p and q. If $k \leqq 3$, any map of G suffices. If $k \geqq 4$, we have a multiple split. Since the given group Σ is admissible, no parallel permutation is allowed; hence any σ with $\sigma p = p$, $\sigma q = q$ must have $\sigma A_i = A_i$ for $i = 1, \cdots, k$. Therefore such a σ is the identity. The only possible $\sigma \neq 1$ must then have $\sigma p = q$, $\sigma q = p$. Any two such automorphs σ and τ would have a product which would be a parallel permutation, hence the identity, so that $\sigma = \tau$ and $\sigma^2 = 1$.

Suppose then that the branch graph G has just one σ with $\sigma p = q$, $\sigma q = p$. If $\sigma A_i = A_i$ holds for all i, any map of G is satisfactory. If, however, $\sigma A_i = A_j$ for some $i \neq j$, the sum $A_i + A_j$ will be a fixed circuit of σ. From the conditions for an admissible group our branch graph could not then contain three fixed edges or vertices, and so there are at most two of the suspended chains with $\sigma A_i = A_i$. By a suitable choice of notation G will consist of the suspended chains

$$A_1, \cdots, A_s, \quad A'_1, \cdots, A'_s, \quad B_1, \; B_2,$$

where either B_1 or B_2 may be absent, and where

$$(1) \qquad \sigma B_1 = B_1, \quad \sigma B_2 = B_2, \quad \sigma A_i = A'_i, \quad \sigma A'_i = A_i, \quad (i = 1, \cdots, s).$$

The map of this branch graph with the c. d. boundaries

$$(2) \qquad B_1 + A_1, \quad B_1 + A'_1, \quad B_2 + A_s, \quad B_2 + A'_s, \quad A_i + A_{i+1}, \quad A'_i + A'_{i+1},$$

for $i = 1, \cdots, s - 1$, will then be a map in which the only automorph σ is extendable.

Finally, suppose that G has ambiguous automorphs θ_i. Since G has no least split, this implies, as in § 6.3, that G must be a circuit. Any map has this circuit twice as a c. d. boundary, and the desired conclusion as to the ambiguous θ's is trivial. Thus Theorem A is established in this case.

7.2. Assume that Theorem A is true for all graphs with fewer than n distinct least splits. Consider a cyclicly connected graph G with an admissible group Σ and with n least splits. By § 4.23, G must have some minimal component. A skeleton G' can be constructed by replacing all the homologs of any one suitable minimal component $M_0 = M$ (see § 4, (2)). The skeleton G' has fewer than n least splits, by Lemma 4.42, and is cyclicly connected by § 4.43. Furthermore the group Σ' on G' is admissible by Lemma 5.4, so that the induction assumption of our Theorem A applies to show that G' has a map in which all automorphs of Σ' are extendable. In this map, the new edge $\mu(p, q)$ used to replace the minimal component M_0 will appear in two c. d. boundaries, which may be written as

$$J = \mu + L, \qquad K = \mu + N, \tag{3}$$

where L and N are two chains in G' with the same ends, p and q, as $\mu = \mu_0$.

Lemma 7.21. *Σ contains no σ of any order twisting M_0.*

For[19] if σ twists M_0, then[20] $\sigma\mu_0 = \mu_0$ and σ leaves the ends p and q fixed. Therefore σ, when suitably extended to the sphere from the map of G', has all the points of μ_0 as fixed points, and hence by Eilenberg's results must have a circuit of fixed points and must therefore be homeomorphic to a reflection of the sphere. This extended σ, and hence σ itself, is of order 2. The group Σ then contains an automorph σ of order 2 twisting M_0, contrary to the first condition for admissibility. The Lemma is established.

We next map $M + \alpha$, where $\alpha(p, q)$ is a new edge with the same ends as M_0. The chief difficulties arise from automorphs ρ in Σ' such that

$$\rho\mu = \mu, \qquad \rho \neq 1, \qquad \rho p = q, \qquad \rho q = p. \tag{4}$$

By Lemma 7.21, there is at most one such ρ. But $\rho^2(M) = M$ and $\rho^2(p) = p$, while M has no twists; hence $\rho^2 = 1$.

If ρ is present, then the group generated by ρ is admissible on $M + \alpha$. For no twists or parallel permutations are present. The complement $G - M$

[19] This lemma could also be proven by the remark at the end of § 5.

[20] Henceforth we omit the primes distinguishing the automorphs σ in Σ from the σ' in Σ'.

contains a chain L with ends p and ρp. One argues then, as in § 6, that $L + \rho L$ contains either a fixed edge, vertex or circuit. Therefore, the fixed edge α of $M + \alpha$ corresponds to at least one fixed point of G under ρ. Every fixed point of $M + \alpha$ yields at least one fixed point of G. Thence one concludes that ρ, which is admissible on G, must be admissible on $M + \alpha$.

By § 7. 1 there is a map of $M + \alpha$ in which ρ, if present, is extendable. Denote the c. d. boundaries of this map by

$$(5) \qquad A + \alpha, \qquad B + \alpha, \qquad C_1, \cdots, C_s,$$

where only $A + \alpha$ and $B + \alpha$ contain α. Here the chains A and B can intersect [21] only at their ends p and q. Because ρ is extendable, it preserves the c. d. boundaries (5), so that ρA is A or B. This yields the alternatives

$$(6_1) \qquad \rho A = B,\ \rho B = A, \quad A + B \text{ a fixed circuit of } \rho,$$
$$(6_2) \qquad \rho A = A,\ \rho B = B, \quad A + B \text{ contains two fixed points of } \rho;$$

for in the second alternative A and B are transformed end for end and so each contains a fixed edge or vertex.

On the other hand, since ρ preserves the c. d. boundaries (3) of the map of G', we must have one of the following:

$$(7_1) \qquad \rho J = K, \qquad \rho K = J, \qquad \rho L = N, \qquad \rho N = L,$$
$$(7_2) \qquad \rho J = J, \qquad \rho K = K, \qquad \rho L = L, \qquad \rho N = N.$$

7. 3. Lemma. *If ρ of (4) is present, then either (6_1) and (7_1) hold together, or both (6_2) and (7_2) hold; provided that, if ρ is ambiguous in G', the map of G' is so chosen that $\chi(\rho)$ has a prescribed value ± 1.*

Proof. Suppose first that in the given map of G' there exists a fixed edge or vertex x contained in both L and N. On the map of G' let the point X be x or the midpoint of x and let the point Y be the midpoint of the other fixed edge μ. Join X to Y by two arcs β and γ lying in the domains bounded respectively by J and K. It follows that $\beta + \gamma$ is a circuit cutting G' into two pieces, one inside $\beta + \gamma$ and containing p, and one outside and containing q. But $\beta + \gamma$ meets G' only at the vertices X and Y invariant under ρ. Because $\rho p = q$, ρ must interchange the two pieces of G' which are cut apart at X and Y. Therefore, a combinatorial fixed point of G' under ρ can be in neither of these pieces and so must be at either X or Y. Hence ρ is ambiguous on the edge μ in G' according to the definition of § 6. 1.

[21] This fact was proved in Lemma 5. 41, if M is least, while it is obvious, if M is a branch graph.

By the induction assumption the map of G' can therefore be chosen so that $\chi(\rho)$ is $+1$ or -1; that is, so that either (7_1) or (7_2) holds. If the choice is regulated by the presence of (6_1) or (6_2), we get the conclusion of our Lemma.

Consider now the other case when L and N have no fixed edges or vertices in common. Either (7_1) or (7_2) holds in the given map of G'. If (7_2) holds, L and N must each contain a fixed edge or vertex, so that we have two distinct fixed points x' and y'. By the argument of § 5. 42, ρ has corresponding fixed points x and y in $G-M$. If M should then, as in (6_1), contain the fixed circuit $z=A+B$ under ρ, we would have three fixed points x, y, and z under the automorph ρ of order 2, while x and y lie in the combinatorial side $G-M$ of z. This contradicts the condition (ii) of our Theorem I as restated in the definition of an admissible group. Hence (7_2) implies (6_2), as in the Lemma.

Next suppose that (7_1) holds. If L has a vertex t with $\rho t \in L$, then $\rho t \neq t$, for otherwise ρt would be in $\rho L = N$, and thus would be a common fixed point as in the case previously treated. As in a previous argument we construct in L a subchain L^* (which may equal L), with ends r and s, such that $\rho r = s$ while L^* contains no interior vertex t with $\rho t \in L^*$. L^* cannot be a single fixed edge, for such an edge would be a common fixed point of L and N. Therefore $L^* + \rho L^*$ is a circuit which is a combinatorial fixed point under ρ. If, as in (6_2), M should then contain two fixed points y and z under ρ, we would again find a contradiction to the definition of admissibility. Thus (6_2) is impossible, so (7_1) implies (6_1), as required. The Lemma is now established.

7. 4. We now combine the maps of G' and $M=M_0$. Since all k of the minimal components M_i are homologous, there must be automorphs

$$\tau_0 = 1, \tau_1, \tau_2, \cdots, \tau_{k-1}$$

in the group Σ such that

$$(8) \qquad \tau_i\mu_0 = \mu_i, \qquad \tau_i M_0 = M_i, \qquad (i = 0, 1, \cdots, k-1).$$

Any σ in the group carries M_0 to some M_i, so that $\tau_i^{-1}\sigma$ sends M_0 to M_0. Hence any σ has one of the forms

$$(9) \qquad \sigma = \tau_i, \qquad \sigma = \tau_i\rho,$$

and therefore either the automorphs τ_i or these automorphs together with ρ generate the whole group.

In the map of the skeleton G', $\mu = \mu_0$ is on two c. d. boundaries J and K. Introduce the notation

$$(10) \qquad J_i = \tau_i J, \qquad K_i = \tau_i K, \qquad A_i = \tau_i A, \qquad B_i = \tau_i B, \qquad (i = 1, \cdots, k-1).$$

Obviously, μ_i is on the boundaries J_i and K_i in the map of G'. Furthermore any τ_i applied to the map of $M_0 + \alpha$ in (5) gives a map of the corresponding $M_i + \alpha_i$ with the c. d. boundaries

$$(11) \qquad A_i + \alpha_i, \qquad B_i + \alpha_i, \qquad \tau_i C_1, \cdots, \tau_i C_s,$$

where α_i is a new edge with the same ends as M_i. Maps of G can be constructed by superimposing on the map of the skeleton these maps of the minimal components. By Lemma 2.4 this can be done so that μ_i, formerly on the boundaries J_i and K_i, is so replaced by M_i that the chain A_i on the "rim" of M_i falls into the domain bounded by J_i, while the other chain B_i falls into the domain bounded by K_i. In other words, each boundary C of the map of G' yields in the map of G a boundary C^*, obtained from C by replacing each edge μ_i in C as follows:

$$(12) \qquad \text{Replace } \mu_i \text{ by } A_i \text{ if } C = J_i; \quad \text{by } B_i \text{ if } C = K_i.$$

These circuits C^*, together with the circuits $\tau_i C_j$, are the c. d. boundaries of the map of G. This rule (12) is unambiguous except when $J_i = K_i$ for some i. This can happen only when G' is a circuit. Hence *assume for the present that G' is not a circuit.* (See, however, § 8.)

Since ρ leaves fixed the original set (5) of boundaries C_i, it follows readily from (9) that any σ of the group will carry a c. d. boundary of the form $\tau_i C_j$ into another circuit of the same form. It remains to show that the circuits $\sigma(C^*)$ are also c. d. boundaries. We shall prove more explicitly that

$$(13) \qquad \sigma(C^*) = (\sigma C)^*,$$

where $(\sigma C)^*$ is certainly a c. d. boundary. For (13) it suffices to show that when μ_i is replaced by A_i in the construction (12), then $\sigma\mu_i$ is replaced by σA_i, and similarly for B_i. But for given σ and i there is a j so that $\sigma\mu_i = \mu_j$. For this σ, i, and j we need only show that one of the alternatives

$$(14_1) \qquad \sigma A_i = A_j \quad \text{and} \quad \sigma J_i = J_j,$$

$$(14_2) \qquad \sigma A_i = B_j \quad \text{and} \quad \sigma J_i = K_j$$

must hold. We know already that $\sigma\mu_i = \mu_j$, $\sigma M_i = M_j$ and hence by (9), (10), (6), and (7) that σA_i is either A_j or B_j and that σJ_i is either J_j or K_j.

Case 1. The group does not contain a ρ. Then $\sigma\tau_i$ carries μ_0 to μ_i to μ_j, hence $\sigma\tau_i = \tau_j$. Consequently,

$$(15) \qquad \begin{aligned} \sigma A_i &= \sigma\tau_i A_0 = \tau_j A_0 = A_j, \\ \sigma J_i &= \sigma\tau_i J_0 = \tau_j J_0 = J_j, \end{aligned}$$

by the choice of notation in (10). The first alternative (14_1) holds, as required.

Case 2. Both (6_1) and (7_1) hold, as in Lemma 7.3. Then

$$\rho A = B, \qquad \rho B = A, \qquad \rho J = K, \qquad \rho K = J. \tag{16}$$

Now consider (14) for any σ and i. Since $\sigma\tau_i$ carries μ_0 to μ_j, $\sigma\tau_i$ is either τ_j or $\tau_j\rho$ by (9). If $\sigma\tau_i = \tau_j$, then (15) holds again, as required in (14). Alternatively, if $\sigma\tau_i = \tau_j\rho$ holds, then (16) and (10) entail

$$\sigma A_i = \sigma\tau_i A_0 = \tau_j\rho A_0 = \tau_j B_0 = B_j,$$
$$\sigma J_i = \sigma\tau_i J_0 = \tau_j\rho J_0 = \tau_j K_0 = K_j,$$

which is the second alternative of (14).

Case 3. Both (6_2) and (7_2) hold, as in Lemma 7.3. Then

$$\rho A = A, \qquad \rho B = B, \qquad \rho J = J, \qquad \rho K = K.$$

From this point we obtain the result (14) much as in the previous case.

Thus in all cases we have established one of (14_1) and (14_2), so that the c. d. boundaries C^* of our map of G introduced in (12) are in fact carried into boundaries by any σ. Thus all the automorphs σ of the group Σ are extendable.

7.5. Now let Σ contain automorphs $\theta_1, \theta_2, \cdots, \theta_g$, no two of them conjugate, and each θ_i ambiguous on an edge β_i of G, and let g numbers $\epsilon_i = \pm 1$ be given. By § 6.4, each β_i is also an edge of the skeleton G', so that each θ_i is ambiguous on the corresponding β_i in G'. Hence our induction assumption yields a map of G' in which $\chi(\theta_i) = \epsilon_i$ for $i = 1, \cdots, g$. If the "reflection" ρ should be present and ambiguous on G', then $\rho(M_0) = M_0$ implies that ρ is not ambiguous on G, so that ρ cannot be conjugate to any of the ambiguous θ's (see § 6.6). Thus the map of G' with $\chi(\theta_i) = \epsilon_i$ can also be chosen so that $\chi(\rho)$ has an assigned value, and hence so that Lemma 7.3 applies. Construct the corresponding map of G as in (12). The definition of χ and the relation (13) together imply that $\chi(\theta_i)$ has the same value in this map of G as it does in the map of G'. Thus we have an "extendable" map of G with the assigned characters $\chi(\theta_i) = \epsilon_i$, and the induction is complete in this case.

8. The circular case. It remains only to consider the omitted case of § 7.4 in which the skeleton G' is a circuit. Choose the notation so that the new edges are $\mu_i = \mu_i(p_i, q_i)$, where [22] the vertices $p_i\, q_i$ are in the cyclic order

$$p_0q_0p_1q_1p_2q_2 \cdots p_{k-1}q_{k-1} \tag{1}$$

[22] Throughout this section, subscripts i, j, etc. run from 0 to $k-1$, or may be reduced mod k.

on the circuit G'. In addition to the μ_i, G' contains chains $\nu_i = \nu_i(q_i, p_{i+1})$. Any automorph carries μ's to μ's, hence carries ν's to ν's, so that without altering Σ we can assume that each ν_i is a single edge.[23] Thus G' has the edges $\mu_0, \nu_0, \mu_1, \nu_1, \cdots, \mu_{k-1}, \nu_{k-1}$, in that order, and

$$G = M_0 + \nu_0 + M_1 + \nu_1 + \cdots + M_{k-1} + \nu_{k-1}, \qquad \nu_i = \nu_i(q_i, p_{i+1}),$$

where the subgraphs on the right have no common edges and only the vertices (1), as indicated, in common.

Every automorph σ of G' carries each μ_i into some μ_j and preserves or inverts the order (1). Hence every σ belongs to the dihedral group generated by ω and ρ, where ω is a "rotation" of $2\pi/k$ radians and ρ a "reflection" in the "diameter" through ν_0:

$$(2) \qquad \omega p_i = p_{i+1}, \quad \omega q_i = q_{i+1} \quad \omega\mu_i = \mu_{i+1}, \quad \omega\nu_i = \nu_{i+1};$$

$$(3) \qquad \rho p_i = q_{k+1-i}, \quad \rho q_i = p_{k+1-i}, \quad \rho\mu_i = \mu_{k+1-i}, \quad \rho\nu_i = \nu_{k-i}.$$

One readily obtains the following relations:

$$(4) \qquad \omega^k = 1, \quad \rho^2 = 1, \quad \rho\omega^i = \omega^{-i}\rho, \quad \omega^i\rho = \rho\omega^{-i}.$$

The group Σ' must contain automorphs taking μ_0 to each μ_i. The only σ's with $\sigma(\mu_0) = \mu_1$ are $\sigma = \rho$ and $\sigma = \omega$. If ω is absent and $k > 2$, then the only σ's with $\sigma(\mu_0) = \mu_2$ are ω^2 and $\rho\omega^{k-1} = \omega\rho$. But $\omega\rho$ and ρ together give ω. Hence Σ' must contain ω or else must contain ρ and ω^2. The only such groups Σ' are the three following ones, where each group is determined by the generators given in braces:

$$(5) \qquad \Sigma_1\colon \{\omega^2, \rho\}, \text{ with } k = 2t; \quad \Sigma_2\colon \{\omega\}; \quad \Sigma_3\colon \{\omega, \rho\}.$$

For Σ_3, note that $\rho\omega(\mu_0) = \mu_0$, so that $\rho\omega$ leaves μ_0 fixed.

To map G, first take a new edge $\alpha(p_0, q_0)$ and choose a map of $M_0 + \alpha$ as in § 7, (5). If the group is Σ_3, then $\rho\omega$ is an automorph of $M + \alpha$ if we set $\rho\omega(\alpha) = \alpha$. In this case the map of $M + \alpha$ is to be chosen, as in § 7.2, so that $\rho\omega$ is extendable on the map. Choose automorphs $\tau_i \in \Sigma'$ such that $\tau_i\mu_0 = \mu_i$; if ω is present, choose $\tau_i = \omega^i$ and apply § 7, (10). Then $M_i + \alpha_i$, where $\alpha_i = \alpha_i(p_i, q_i)$, has a map with the c. d. boundaries of § 7, (11). Superimpose all these maps on G' so that the A's all fall on one side of G'; that is, so that G has a map with the c. d. boundaries, $\tau_i C_j$, for $i = 0, \cdots, k-1$, and $j = 1, \cdots, s$, and also two boundaries

$$(6) \qquad D_1 = A_0 + \nu_0 + A_1 + \nu_1 + \cdots + A_{k-1} + \nu_{k-1},$$

$$(7) \qquad E_1 = B_0 + \nu_0 + B_1 + \nu_1 + \cdots + B_{k-1} + \nu_{k-1}.$$

[23] If $q_i = p_{i+1}$, ν_i is a single vertex. The formulas below will apply also to this case.

If the group is Σ_1 or Σ_2, the τ_i include the whole group, and every τ_i is extendable, so that we have a map with all automorphs extendable.

If the group is Σ_3, $\tau_i = \omega^i$, $\rho\omega(\mu_0) = \mu_0$, and $\rho\omega$ is extendable on $M_0 + \alpha$, so that $\rho\omega(A_0)$ is A_0 or B_0. Therefore one of the alternatives

$$(8_1) \qquad \rho A_0 = A_1, \qquad \rho B_0 = B_1;$$

$$(8_2) \qquad \rho A_0 = B_1, \qquad \rho B_0 = A_1;$$

must hold.[24] In the first alternative, (4) and the definition $\tau_i = \omega^i$ show that

$$(9_1) \qquad \rho A_i = \rho\omega^i A_0 = \omega^{-i}\rho A_0 = \omega^{-i} A_1 = A_{1-i}, \qquad \rho B_i = B_{1-i}.$$

In the second alternative, one similarly calculates that

$$(9_2) \qquad \rho A_i = B_{1-i}, \qquad \rho B_i = A_{1-i}.$$

It follows that ρ interchanges D_1 and E_1 or leaves them each fixed according as (8_2) or (8_1) holds. But $\rho\omega$ also leaves the set of all C_s fixed, because the map of $M_0 + \alpha$ was picked so that $\rho\omega$ is extendable on this map. Hence $\rho(\tau_i C_j) = \omega^{1-i}(\rho\omega C_j)$ is a c. d. boundary, and ρ is extendable in the map of G. But ω is extendable by the construction of (6) and (7), so all σ's in Σ_3 are extendable.

An automorph θ of G could be ambiguous only on some edge ν_j, by § 6. 4. But $\rho\omega^{k-2j}$ is the only $\sigma \neq 1$ with ν_j as a fixed edge. Hence the ambiguous θ's are all in the set

$$(10) \qquad \theta_j = \rho\omega^{k-2j}, \qquad (j = 0, 1, \cdots, k-1).$$

Every $\theta_j^2 = 1$. If k is odd, then $k = 2t + 1$, $\theta_j(\mu_{t+j+1}) = \mu_{t+j+1}$ by (2) and (3), and θ_j interchanges the ends of M_{t+j+1}. As in Lemma 5. 41, θ_j must then have a fixed circuit or two fixed points in M_{t+j+1}. These, in addition to the fixed point ν_j, show that θ_j is not ambiguous on G. There remains only the case $k = 2t$, in which event θ_j is ambiguous on both ν_j and ν_{j+t} in G.

Possible conjugates of θ_j are $\omega^i\theta_j\omega^{-i} = \theta_{i+j}$ and $\rho\theta_j\rho^{-1} = \theta_{t-j}$. Therefore all θ's are conjugate in Σ_3, while in Σ_2 of (5) there are no θ's present. For the group Σ_1 of (5), θ_0 has conjugates $\theta_0, \theta_2, \theta_4, \cdots$ and $\theta_t, \theta_{t-2}, \cdots$, where $k = 2t$. Thus all θ's are conjugates if t is odd, while if t is even θ_0 and θ_1 are not conjugates, but every θ is conjugate to one of θ_0 or θ_1.

For alternative maps when $k = 2t$, we again superimpose the maps of $M_i + \alpha_i$ on G', but now so that the c. d. boundaries are not (6) and (7), but

$$(11) \qquad \begin{aligned} D_2 &= \Sigma\nu_i + A_0 + B_1 + A_2 + B_3 + \cdots + A_{k-2} + B_{k-1}, \\ E_2 &= \Sigma\nu_i + B_0 + A_1 + B_2 + A_3 + \cdots + B_{k-2} + A_{k-1}, \end{aligned}$$

[24] The automorph ρ, though a "reflection" of the circuit G', is not necessarily a "reflection" of G.

where $\Sigma\nu_i$ is the subgraph of all the ν_i. If $k = 2t$, and t is even, use

$$(12)\quad \begin{aligned} D_3 &= \Sigma\nu_i + A_0 + A_1 + B_2 + B_3 + A_4 + A_5 + B_6 + B_7 + \cdots + B_{k-2} + B_{k-1} \\ E_3 &= \Sigma\nu_i + B_0 + B_1 + A_2 + A_3 + B_4 + B_5 + A_6 + A_7 + \cdots + A_{k-2} + A_{k-1} \end{aligned}$$

as the c. d. boundaries, or else use the circuits D_4 and E_4 obtained from D_3 and E_3 by increasing all subscripts on the right of (12) by 1. Call the latter map (12 +).

If the group is Σ_3, and $k = 2t$, then ω is extendable in either map (7) or map (11). All θ's in (10) are conjugate to $\theta_0 = \rho$. As in (9), ρ carries μ_i to μ_{1-i}, hence takes odd subscripts to even ones. If (9_1) holds, ρ carries A's to A's, so that $\rho(D_2) = E_2$, $\rho(E_2) = D_2$. In both maps ρ is extendable, but, by the definition in § 6. 5, $\chi(\rho)$ is $+1$ in (11), and -1 in (7). If the other alternative (9_2) holds, ρ is again extendable, but $\chi(\rho) = +1$ in (7), and -1 in (11). In either event a map can be picked so that all σ's are extendable, and ρ, the typical ambiguous automorph, has either character desired.

If the group is Σ_1, and $k = 2t$, similar calculations can be made. By (10), (2), and (3), one has

$$\rho\mu_{2i} = \mu_{1-2i}, \qquad \rho\mu_{2i+1} = \mu_{-2i}, \qquad \theta_1\mu_{2i} = \mu_{3-2i}, \qquad \theta_1\mu_{2i+1} = \mu_{2-2i}.$$

If t is odd, all θ's are conjugate to $\theta_0 = \rho$, and (7) and (11) again give maps with ω^2 and ρ extendable and with $\chi(\rho) = -1$ in (7), and $\chi(\rho) = +1$ in (11). If t is even, all θ's are conjugate to θ_0 or θ_1. But ω^2 and ρ are extendable in all four maps, and the characters $\chi(\rho)$ and $\chi(\theta_1)$ are, in that order, $(-1, -1)$ in (7), $(+1, +1)$ in (11), $(-1, +1)$ in (12), $(+1, -1)$ in (12 +). This gives all four possible sets of characters.

This completes the proof that when the skeleton is a circuit there is a map of G in which all σ's are extendable and in which the characters of ambiguous automorphs are arbitrarily assigned. The induction for Theorem A is finished, and with it the proof of Theorem 1 and its consequences in § 1.

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ASYMPTOTIC CURVES ON A SURFACE.*

By E. P. Lane and M. L. MacQueen.

1. Introduction. The purpose of this paper is to make some contributions to the theory of the asymptotic curves on an analytic non-ruled surface in ordinary projective space. Section 2 contains a summary of portions of the classical analytical theory of the projective differential geometry of curves and surfaces which are used in later developments. Power series expansions in non-homogeneous projective coördinates for the parametric asymptotic curves on the surface are computed to terms of higher degree than have hitherto been considered. By means of these power series a brief study is made of the asymptotic curves regarded as space curves.

2. Analytic basis. If the four homogeneous projective coördinates x of a variable point on an analytic non-ruled surface S in ordinary space are given as analytic functions of two independent variables u, v, and if the parametric net on S is the asymptotic net, then the functions x are solutions of a system of differential equations which can be reduced to the form

$$\begin{aligned} x_{uu} &= px + \theta_u x_u + \beta x_v, \\ x_{vv} &= qx + \gamma x_u + \theta_v x_v \end{aligned} \qquad (\theta = \log \beta\gamma). \tag{1}$$

The coefficients of these equations are functions of u, v and satisfy certain integrability conditions.

The formulas for the third derivatives of x expressed as linear combinations of x, x_u, x_v, x_{uv} are found from system (1) to be

$$\begin{aligned} x_{uuu} &= (p_u + p\theta_u)x + (p + \theta_u^2 + \theta_{uu})x_u + (\beta_u + \beta\theta_u)x_v + \beta x_{uv}, \\ x_{uuv} &= (p_v + \beta q)x + kx_u + \pi x_v + \theta_u x_{uv}, \\ x_{uvv} &= (q_u + \gamma p)x + \chi x_u + kx_v + \theta_v x_{uv}, \\ x_{vvv} &= (q_v + q\theta_v)x + (\gamma_v + \gamma\theta_v)x_u + (q + \theta_v^2 + \theta_{vv})x_v + \gamma x_{uv}, \end{aligned} \tag{2}$$

where π, χ, k are defined by the formulas

$$\pi = p + \beta\psi, \qquad \chi = q + \gamma\phi, \qquad k = \beta\gamma + \theta_{uv}, \tag{3}$$

and ϕ, ψ by

$$\phi = (\log \beta\gamma^2)_u, \qquad \psi = (\log \beta^2\gamma)_v. \tag{4}$$

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Furthermore, we find

$$x_{uuuu} = A_4 x + B_4 x_u + C_4 x_v + D_4 x_{uv}, \tag{5}$$

where

$$\begin{aligned} A_4 &= (p_u + p\theta_u)_u + p(p + \theta_u^2 + \theta_{uu}) + \beta(p_v + \beta q), \\ B_4 &= (p_u + p\theta_u) + \theta_u(p + \theta_u^2 + \theta_{uu}) + (p + \theta_u^2 + \theta_{uu})_u + \beta k, \\ C_4 &= \beta(p + \theta_u^2 + \theta_{uu}) + (\beta_u + \beta\theta_u)_u + \beta\pi, \\ D_4 &= 2(\beta_u + \beta\theta_u). \end{aligned} \tag{6}$$

In fact it is possible to express every u-derivative of x as a linear combination of x, x_u, x_v, x_{uv} by an equation of the form

$$\partial x^i/\partial u^i = A_i x + B_i x_u + C_i x_v + D_i x_{uv}, \tag{7}$$

where

$$\begin{aligned} A_i &= A_{i-1,u} + pB_{i-1} + (p_v + \beta q)D_{i-1}, \\ B_i &= A_{i-1} + \theta_u B_{i-1} + B_{i-1,u} + kD_{i-1}, \\ C_i &= \beta B_{i-1} + C_{i-1,u} + \pi D_{i-1}, \\ D_i &= C_{i-1} + \theta_u D_{i-1} + D_{i-1,u}. \end{aligned} \tag{8}$$

The parametric vector equation of an analytic curve is

$$x = x(t), \tag{9}$$

the parameter being t. These coördinates x satisfy an ordinary differential equation of the form

$$x^{iv} + 4p_1 x''' + 6p_2 x'' + 4p_3 x' + p_4 x = 0 \qquad (x' = dx/dt, \cdots), \tag{10}$$

the coefficients being functions of t. Let P_2, P_3, P_4 be defined by the formulas

$$\begin{aligned} P_2 &= p_2 - p_1^2 - p_1', \\ P_3 &= p_3 - 3p_1p_2 + 2p_1^3 - p_1'', \\ P_4 &= p_4 - 4p_1p_3 - 3p_2^2 + 12p_1^2p_2 - 6p_1^4 - p_1'''. \end{aligned} \tag{11}$$

Then two invariants θ_3, θ_4 of the differential equation are defined by the formulas

$$\theta_3 = P_3 - \tfrac{3}{2}P_2', \quad \theta_4 = P_4 - \tfrac{81}{25}P_2^2 - 2P_3' + \tfrac{6}{5}P_2''. \tag{12}$$

If the differential equation of the form (10) for the asymptotic u-curves on the surface is calculated by means of equations (1), (2), (5), then the coefficients of this equation are found to be given by the formulas

$$
(13)\qquad
\begin{aligned}
2p_1 &= \phi - 3\theta_u, \\
6p_2 &= \phi^2 + \phi_u - 7\theta_u\phi - 4\theta_{uu} + 11\theta_u{}^2 - 2p - \beta\psi, \\
4p_3 &= 7\theta_u\theta_{uu} - 6\theta_u{}^3 - \theta_{uuu} + 6p\theta_u - 2p_u + 5\theta_u{}^2\phi \\
&\qquad - \theta_u\phi^2 - 2p\phi - 2\theta_{uu}\phi - \theta_u\phi_u + \theta_u\beta\psi - \beta k, \\
p_4 &= p^2 + p\beta\psi - 6p\theta_u{}^2 + 2p\theta_{uu} - p\phi_u + 5p\theta_u\phi \\
&\qquad + 5p_u\theta_u - p_{uu} - p\phi^2 - 2p_u\phi - \beta(p_v + \beta q).
\end{aligned}
$$

The invariants θ_3, θ_4 defined by the formulas (12) can be calculated for the u-curves, but we shall omit the writing of these results here.

3. Power series expansions for the asymptotic curves. In ordinary space an analytic curve can be defined by expressing two of the non-homogeneous coördinates of a point on the curve as power series in the third coördinate. Power series expansions for the u-curve can be calculated in the following way.

The coördinates of any point X near the point x and on the u-curve can be represented by Taylor's expansion as power series in the increment Δ_u corresponding to displacement on S from x to the point X along the curve:

$$
(14)\qquad X = x + x_u\Delta u + \tfrac{1}{2}x_{uu}\Delta u^2 + \tfrac{1}{6}x_{uuu}\Delta u^3 + \tfrac{1}{24}x_{uuuu}\Delta u^4 + \cdots.
$$

Expressing each of $x_{uu}, x_{uuu}, \cdots$ as a linear combination of x, x_u, x_v, x_{uv}, we find

$$
X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv},
$$

where the local coördinates $x_1, \cdots, x_4$ of the point X are given by the expansions

$$
(15)\qquad
\begin{aligned}
x_1 &= 1 + \tfrac{1}{2}p\Delta_u{}^2 + \tfrac{1}{6}(p_u + p\theta_u)\Delta_u{}^3 + \tfrac{1}{24}A_4\Delta_u{}^4 + \cdots, \\
x_2 &= \Delta u + \tfrac{1}{2}\theta_u\Delta u^3 + \tfrac{1}{6}(p + \theta_u{}^2 + \theta_{uu})\Delta u^3 \\
&\qquad + \tfrac{1}{24}B_4\Delta_u{}^4 + \tfrac{1}{120}B_5\Delta_u{}^5 + \cdots, \\
x_3 &= \tfrac{1}{2}\beta\Delta u^2 + \tfrac{1}{6}(\beta_u + \beta\theta_u)\Delta u^3 + \tfrac{1}{24}C_4\Delta u^4 \\
&\qquad + \tfrac{1}{120}C_5\Delta u^5 + \tfrac{1}{720}C_6\Delta u^6 + \cdots, \\
x_4 &= \tfrac{1}{6}\beta\Delta u^3 + \tfrac{1}{12}(\beta_u + \beta\theta_u)\Delta u^4 + \tfrac{1}{120}D_5\Delta u^5 \\
&\qquad + \tfrac{1}{720}D_6\Delta u^6 + \tfrac{1}{5040}D_7\Delta u^7 + \cdots.
\end{aligned}
$$

Introducing non-homogeneous coördinates by the definitions

$$
(16)\qquad x = x_2/x_1, \quad y = x_3/x_1, \quad z = x_4/x_1,
$$

we find

$$
\begin{aligned}
x &= \Delta u + \tfrac{1}{2}\theta_u\Delta u + \tfrac{1}{6}(\theta_u{}^2 + \theta_{uu} - 2p)\Delta u^3 \\
&\quad + \tfrac{1}{24}(B_4 - 4p_u - 10p\theta_u)\Delta_u{}^4 \\
&\quad + \tfrac{1}{120}[B_5 - 5A_4 - 10\theta_u(p_u + p\theta_u)]\Delta u^5 + \cdots, \\
y &= \tfrac{1}{2}\beta\Delta u^2 + \tfrac{1}{6}(\beta_u + \beta\theta_u)\Delta u^3 + \tfrac{1}{24}(C_4 - 6\beta p)\Delta u^4 \\
&\quad + \tfrac{1}{120}[C_5 - 10\beta(p_u + p\theta_u) - 10p(\beta_u + \beta\theta_u)]\Delta u^5 \\
&\quad + \tfrac{1}{720}[C_6 - 15\beta A_4 - 20(\beta_u + \beta\theta_u)(p_u + p\theta_u) \\
&\qquad - 15p(C_4 - 6\beta p)]\Delta u^6 + \cdots, \\
z &= \tfrac{1}{6}\beta\Delta u^3 + \tfrac{1}{12}(\beta_u + \beta\theta_u)\Delta u^4 + \tfrac{1}{120}(D_5 - 10\beta p)\Delta u^5 \\
&\quad + \tfrac{1}{720}[D_6 - 20\beta(p_u + p\theta_u) - 30p(\beta_u + \beta\theta_u)]\Delta u^6 \\
&\quad + \tfrac{1}{5040}[D_7 - 21p(D_6 - 10\beta p) - 35\beta A_4 \\
&\qquad - 70(\beta_u + \beta\theta_u)(p_u + p\theta_u)]\Delta u^7 + \cdots.
\end{aligned} \tag{17}
$$

Inverting the first of these series we obtain

$$\Delta u = x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots, \tag{18}$$

where

$$
\begin{aligned}
c_2 &= -\tfrac{1}{2}\theta_u, \qquad c_3 = -\tfrac{1}{6}(\theta_{uu} - 2\theta_u{}^2 - 2p), \\
c_4 &= -\tfrac{1}{24}(\theta_{uuu} - 7\theta_u\theta_{uuu} + 6\theta_u{}^3 + 12p\theta_u - 2p_u + \beta k), \\
c_5 &= -\tfrac{1}{120}\ (\theta_{uuuu} - 11\theta_u\theta_{uuu} + 46\theta_u{}^2\theta_{uu} - 24\theta_u{}^4 - 7\theta_{uu}{}^2 \\
&\quad + 24p(\theta_{uu} - 3\theta_u{}^2 - p) + 20p_u\theta_u - 2p_{uu} \\
&\quad + \beta[k_u + 6k\theta_u - 3k\phi - 4(p_v + \beta q)]).
\end{aligned} \tag{19}
$$

If we substitute the series (18) for Δu in the last two of the series (17) we arrive at *the power series expansions for the asymptotic u-curve*, namely,

$$
\begin{aligned}
y &= \tfrac{1}{2}\beta x^2 - \tfrac{1}{6}\beta\phi x^3 + \tfrac{1}{24}\beta a_4x^4 + \tfrac{1}{120}\beta a_5x^5 + \tfrac{1}{720}\beta a_6x^6 + \cdots, \\
z &= \tfrac{1}{6}\beta x^3 - \tfrac{1}{12}\beta\phi x^4 + \tfrac{1}{120}\beta b_5x^5 \\
&\qquad + \tfrac{1}{720}\beta b_6x^6 + \tfrac{1}{5040}\beta b_7x^7 + \cdots,
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
a_4 &= \phi^2 + \theta_u\phi - \phi_u + 4p + \beta\psi, \\
a_5 &= a_{4u} - (2\theta_u + \phi)a_4 - 10p\phi - 2\beta\phi\psi - 4\beta k, \\
a_6 &= a_{5u} - (3\theta_u + \phi)a_5 + 3(6p + \beta\psi)a_4 \\
&\qquad + 20\beta\phi k - 2\pi\beta\psi + 10\beta(p_v + \beta q), \\
b_5 &= 3a_4 - 2\beta\psi, \\
b_6 &= 4a_5 + 10\beta\phi\psi + 2\beta(k + \phi_v - 3\theta_{uv}), \\
b_7 &= 5a_6 - (3\theta_u + \phi)(b_6 - 4a_5) \\
&\qquad + b_{6u} - 4a_{5u} - 4\beta\psi(b_5 + 10p) - 10\beta\phi k.
\end{aligned} \tag{21}
$$

Analogous expansions for the v-curve can be written by making the appropriate symmetrical interchanges of the symbols.

4. A canonical form for the expansions. By suitable choice of the coördinate system the power series expansions (20) can be reduced to a simple canonical form which it is the purpose of this section to obtain.

First of all, *the osculating twisted cubic* at the point $O(0,0,0)$ of the u-curve defined by equations (20) is found to have the parametric equations

$$x_1 = 1 + Ct^2 + Dt^3, \qquad x_3 = \tfrac{1}{2}\beta t^2, \tag{22}$$
$$x_2 = t + \tfrac{1}{6}\phi t^2 + Ht^3, \qquad x_4 = \tfrac{1}{6}\beta t^3,$$

wherein C, H, D are defined by the formulas

$$\begin{aligned} C &= \tfrac{1}{60}(3a_4 - 5\phi^2 - 12\beta\psi), \\ H &= -\tfrac{1}{180}(3a_4 - 5\phi^2 + 18\beta\psi), \\ D &= \tfrac{1}{60}a_5 + \tfrac{1}{12}\phi a_4 - \tfrac{2}{27}\phi^3. \end{aligned} \tag{23}$$

Incidentally, it will be observed that C, H satisfy the relation

$$C + 3H = -\tfrac{1}{2}\beta\psi. \tag{24}$$

The osculating twisted cubic may also be written in the form of two power series expansions which must agree with the series (20) up to and including the fifth powers of x. For this result we find

$$\begin{aligned} y &= \tfrac{1}{2}\beta x^2 - \tfrac{1}{6}\beta\phi x^3 + \tfrac{1}{24}\beta a_4 x^4 + \tfrac{1}{120}\beta a_5 x^5 + \tfrac{1}{720}\beta \bar{a}_6 x^6 + \cdots, \\ z &= \tfrac{1}{6}\beta x^3 - \tfrac{1}{12}\beta\phi x^4 + \tfrac{1}{120}\beta b_5 x^5 \\ &\qquad + \tfrac{1}{720}\beta \bar{b}_6 x^6 + \tfrac{1}{5040}\beta \bar{b}_7 x^7 + \cdots, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \bar{a}_6 &= -7\phi a_5 - \tfrac{140}{9}\phi^4 - 140(C - 2H)\phi^2 \\ &\qquad + 360(2C^2 - 8CH + 7H^2), \\ \bar{b}_6 &= 4a_5 + 10\beta\phi\psi, \\ \bar{b}_7 &= 5a_6 - \tfrac{140}{3}\beta\phi^2\psi - 60\beta\psi(5C - 7H). \end{aligned} \tag{26}$$

Let us make the transformation

$$\begin{aligned} x &= (X + \tfrac{1}{6}\phi Y + HZ)/(1 + CY + DZ), \\ y &= (\tfrac{1}{2}\beta Y)/(1 + CY + DZ), \\ z &= (\tfrac{1}{6}\beta Z)/(1 + CY + DZ), \end{aligned} \tag{27}$$

from the old coördinates x, y, z to new coördinates X, Y, Z. This transformation moves the vertex $(0,0,0,1)$ of the tetrahedron of reference to a point on the osculating twisted cubic of the u-curve. Furthermore, the edge $X_1 = X_2 = 0$ of the new tetrahedron is tangent to the osculating cubic at the point $(0,0,0,1)$, and the face $X_1 = 0$ is the osculating plane of the cubic

at this point. The unit point is somewhere on the osculating cubic whose equations become

(28) $$Y = X^2, \qquad Z = X^3.$$

Moreover, we find that *the equations* (20) *of the u-curve can be written in the canonical form*

(29) $$\begin{aligned} Y &= X^2 + A_6 X^6 + \cdots, \\ Z &= X^3 + B_6 X^6 + B_7 X^7 + \cdots, \end{aligned}$$

the coefficients A_6, B_6, B_7 *being defined by the formulas*

(30) $$\begin{aligned} A_6 &= \tfrac{1}{360}(a_6 - \bar{a}_6), \\ B_6 &= \tfrac{1}{120}(b_6 - \bar{b}_6), \\ B_7 &= \tfrac{1}{840}(b_7 - \bar{b}_7 + \phi B_6). \end{aligned}$$

If we replace $\bar{b}_6$ in the second of equations (30) by its value found in equations (26), we find

(31) $$B_6 = \tfrac{1}{60}\beta[\beta\gamma - (\log\beta)_{uv}].$$

The coefficient B_6 is found to be an invariant and differs from the invariant θ_3 of the differential equation of the u-curve only by a constant factor. In fact, if θ_3 is calculated, it is found that

(32) $$B_6 = -\tfrac{1}{15}\theta_3.$$

Moreover, further calculation shows that

(33) $$A_6 = -\tfrac{1}{180}(5\theta_4 + 2\theta'_3), \qquad B_7 = -\tfrac{1}{420}(25\theta_4 + 14\theta'_3),$$

so that we have

(34) $$B_7 - 3A_6 = \tfrac{1}{42}\theta_4.$$

Therefore *the expansions for the u-curve may be written in the form*

(35) $$\begin{aligned} Y &= X^2 - \tfrac{1}{180}(5\theta_4 + 2\theta'_3)X^6 + \cdots, \\ Z &= X^3 - \tfrac{1}{15}\theta_3 X^6 - \tfrac{1}{420}(25\theta_4 + 14\theta'_3)X^7 + \cdots. \end{aligned}$$

5. Asymptotic curves regarded as space curves. Application of the theory of space curves to the power series expansions of the asymptotic curves leads to some interesting results. In the old coördinate system the equations of the osculating quadric cones at the point O of the u-curve and the v-curve are respectively

(36) $$\begin{aligned} y^2 - (\tfrac{3}{2}\beta x - \tfrac{1}{2}\phi y - 9H^{(u)}z)z &= 0, \\ x^2 - (\tfrac{3}{2}\gamma y - \tfrac{1}{2}\psi x - 9H^{(v)}z)z &= 0, \end{aligned}$$

where $H^{(u)}$ is the same as the H defined in equations (23) and $H^{(v)}$ is easily written therefrom. The polar plane of the v-tangent, $x = z = 0$, with respect to the osculating quadric cone of the u-curve is the plane

$$y + \tfrac{1}{4}\phi z = 0. \tag{37}$$

Similarly, the polar plane of the u-tangent, $y = z = 0$, with respect to the osculating quadric cone of the v-curve is

$$x + \tfrac{1}{4}\psi z = 0. \tag{38}$$

Thus *the polar plane of the u-tangent with respect to the osculating quadric cone of the v-curve intersects the polar plane of the v-tangent with respect to the osculating quadric cone of the u-curve in the first edge of Green.*

Another quadric cone can be obtained from the two osculating quadric cones of the asymptotic curves. For this purpose let us consider any line l through the point O and lying in the tangent plane to the surface at the point O. This line has the equations

$$y - nx = z = 0 \qquad (n \neq 0). \tag{39}$$

The polar planes of the line l with respect to the cones (36) are respectively

$$y + 1/4n(n\phi - 3\beta)z = 0 \quad \text{and} \quad x + \tfrac{1}{4}(\psi - 3n\gamma)z = 0. \tag{40}$$

Eliminating n from these equations, we find

$$(x + \tfrac{1}{4}\psi z)(y + \tfrac{1}{4}\phi z) - \tfrac{9}{16}\beta\gamma z^2 = 0. \tag{41}$$

The following theorem summarizes this result:

Consider any line l through the point O of a surface and lying in the tangent plane of the surface at the point. Consider the polar planes of the line l with respect to the two osculating quadric cones of the asymptotic curves through the point O. As l varies about the point O the line of intersection of these two planes generates a quadric cone with its vertex at the point O.

It may be remarked that Calapso has obtained[1] the cone (41) by a method based on a consideration of the osculating twisted cubics of the asymptotic curves at the point 0 of the surface. He shows, among other things, that the polar line of the tangent plane of the surface at the point 0 with respect to this cone is the first edge of Green.

---

[1] R. Calapso, "Sugli enti proiettivi legati al generico punto di una superficie," *Atti Accademia Gioenia*, Catania, (5), vol. 19 (1932), mem. 14.

From the canonical expansions (29) we find that the u-curve belongs to a linear complex in case $B_6 = 0$, and is a twisted cubic in case $B_6 = A_6 = 0$. In the new coördinate system the principal plane at the point O of the u-curve is found to have the equation

$$B_6 Y - A_6 Z = 0, \tag{42}$$

and the principal point on the u-tangent has the coördinates

$$(3A_6,\ B_6,\ 0,\ 0). \tag{43}$$

If the u-curve belongs to a linear complex it is obvious that the principal plane coincides with the osculating plane of the curve and that the principal point coincides with the point O. Moreover, the principal plane is indeterminate in case the u-curve is a twisted cubic.

The Halphen cone at the point O of the u-curve is the eleven-point cubic cone with its vertex at the point O and with the u-tangent for a double line. In the new coördinate system *the equation of the Halphen cone at the point O of the u-curve is found to be*

$$X\left(YZ - \frac{B_7 - 2A_6}{B_6} Z^2\right) - Y^3 + \left(\frac{B_7 - 2A_6}{B_6}\right) Y^2 Z - B_6 Z^3 = 0. \tag{44}$$

The nodal plane of this cone has the equation

$$B_6 Y - (B_7 - 2A_6) Z = 0. \tag{45}$$

This plane coincides with the principal plane defined by equation (42) if, and only if,

$$B_7 - 3A_6 = 0. \tag{46}$$

Referring to equation (34), we thus obtain the following geometrical interpretation of the vanishing of the invariant θ_4:

The principal plane at a point of the u-curve coincides with the nodal plane of the Halphen cone at the point if, and only if, $\theta_4 = 0$.

Another geometrical interpretation for the condition $\theta_4 = 0$ can be given in the following way. The plane containing the three inflexional generators of the Halphen cone is found to have the equation

$$B_6^2 X - 2B_6(B_7 - 2A_6) Y + (B_7 - 2A_6)^2 Z = 0. \tag{47}$$

The pencil of quadric surfaces containing the osculating twisted cubic and having seven-point contact with the u-curve at the point 0 is given by

$$h[Y - X^2 - (A_6/B_6)(Z - XY)] + k(Y^2 - ZX) = 0, \tag{48}$$

where h, k are arbitrary constants. Among these quadrics there are only two cones, for which

$$h^2(kB_6^2 + hA_6^2)^2 = 0. \tag{49}$$

The one of these for which $h = 0$ is the osculating quadric cone

$$Y^2 - ZX = 0 \tag{50}$$

with its vertex at the point O. The other, for which

$$kB_6^2 + hA_6^2 = 0,$$

has the equation

$$B_6^2(Y - X^2) - A_6B_6(Z - XY) - A_6^2(Y^2 - ZX) = 0, \tag{51}$$

and its vertex is at the Halphen point

$$(A_6^3,\ A_6^2B_6,\ A_6B_6^2,\ B_6^3). \tag{52}$$

The polar plane of the principal point (43) with respect to the cone (51) is found to have the equation

$$B_6^2X - 2A_6B_6Y + A_6^2Z = 0. \tag{53}$$

Comparison of this equation and equation (47) shows that these two planes coincide in case $B_7 - 3A_6 = 0$. Thus the following theorem is proved:

At a point O of the u-curve the plane containing the three inflexional generators of the Halphen cone coincides with the polar plane of the principal point with respect to the cone defined by equation (51) *if, and only if,* $\theta_4 = 0$.

Finally, it is easy to establish the following theorem:

If the u-curve belongs to a linear complex, the Halphen cone is composite, one component being the osculating plane and the other being the osculating quadric cone. Moreover, the osculating quadric cone has eight-point contact with the curve, provided $B_7 - 2A_6 \neq 0$.

THE UNIVERSITY OF CHICAGO.

SYSTEMS OF QUADRICS ASSOCIATED WITH A POINT OF A SURFACE.*

By Louis Green.

1. Introduction. One of the popular methods of studying the local geometry of an analytic surface immersed in a three-dimensional projective space is to examine the properties of algebraic osculants. For, these osculants can be used to obtain simple characterizations of many elements intimately related to the given surface. The osculating quadrics at a point of a surface are of particular importance in this respect, and it is the purpose of this paper to make a more comprehensive study of certain of these quadrics, obtaining as a result significant properties of the surface.

Among other things, the problem of determining the envelope of a two-parameter family of Moutard quadrics is solved, the singularity of the curve of intersection of the surface and one of its Moutard quadrics is investigated, certain individual members of each Moutard pencil of quadrics [1] are characterized, and three new pencils of quadrics which we call the Darboux-Segre pencils of quadrics are defined and discussed.

2. Analytic basis. The local coördinate tetrahedron of G. M. Green (with a modified unit point) is taken as the projective coördinate system throughout this paper, and the equations of the various configurations to be used are obtained in this section by transformation from Fubini's coördinate system.

Fubini's canonical differential equations of an analytic curved surface S are given by

$$x_{uu} = px + \theta_u x_u + \beta x_v, \qquad x_{vv} = qx + \gamma x_u + \theta_v x_v, \qquad (\theta = \log \beta\gamma).$$

Let 0 be an ordinary point of S, chosen as the vertex $(1, 0, 0, 0)$ of two local tetrahedrons, one of Fubini and the other of Green. If a point of the ambient three-space has the two sets of coördinates $(x_1, \cdots, x_4)$, $(X_1, \cdots, X_4)$ relative to these respective tetrahedrons, then the following relations exist: [2]

* Presented to the Society, April 9, 1937. Received by the Editors October 30, 1937.

[1] So called because each of these pencils contains a quadric of Moutard. These pencils were first studied by Su and Ichida, *Japanese Journal of Mathematics*, vol. 10 (1933), pp. 209-216.

[2] Lane, "Power series expansions in the neighborhood of a point on a surface," *Proceedings of the National Academy of Sciences*, vol. 13 (1927), pp. 808-813.

$$
(1)\quad
\begin{aligned}
x_1 &= 4X_1 + 3l\phi X_2 + 3m\psi X_3 + (9/4)n(\phi\psi - 8\theta_{uv})X_4,\\
x_2 &= -12lX_2 - 9n\psi X_4,\\
x_3 &= -12mX_3 - 9n\phi X_4,\\
x_4 &= 36nX_4,
\end{aligned}
$$

where l, m, n, ϕ, ψ, have the values

$$
\begin{aligned}
l &= 1/(\beta^2\gamma)^{1/3}, \qquad m = 1/(\beta\gamma^2)^{1/3}, \qquad n = lm,\\
\phi &= (\log \beta\gamma^2)_u, \qquad \psi = (\log \beta^2\gamma)_v.
\end{aligned}
$$

With the introduction of non-homogeneous coördinates X, Y, Z defined in the customary way, 0 becomes the origin of a local non-homogeneous coördinate system, and the coördinates of a point on S sufficiently "near" 0 satisfy a convergent power series expansion of the form

$$
(2)\quad
\begin{aligned}
Z = XY + X^3 + Y^3 + AX^3Y + BXY^3 + C_0X^5 + C_1X^4Y\\
+ C_2X^3Y^2 + C_3X^2Y^3 + C_4XY^4 + C_5Y^5 + \cdots,
\end{aligned}
$$

where all the coefficients are absolute invariants of the surface, and in particular,

$$
(3)\quad
\begin{aligned}
A &= -3m\psi/2, \qquad B = -3l\phi/2,\\
20C_0 &= 36l^2p - 9l^2\phi_u - 27m\psi - (9/4)l^2\phi^2 + 9l^2\phi\theta_u,\\
16C_1 &= 24n\psi_u - 12n\phi_v + 135n\phi\psi - 135,\\
16C_2 &= -135m^2q + 72m^2\psi_v + 27m^2\psi^2 - 180l\phi + 72m^2\psi(\beta_v/\beta),\\
16C_3 &= -135l^2p + 72l^2\phi_u + 27l^2\phi^2 - 180m\psi + 72l^2\phi(\gamma_u/\gamma),\\
16C_4 &= 24n\phi_v - 12n\psi_u + 135n\phi\psi - 135,\\
20C_5 &= 36m^2q - 9m^2\psi_v - 27l\phi - (9/4)m^2\psi^2 + 9m^2\psi\theta_v.
\end{aligned}
$$

For use in § 7 the formulas for the derivatives of the local coördinates $X_1, \cdots, X_4$ will now be obtained. Equations (1) are first solved for the X_i, the resulting equations are then differentiated with respect to $u(v)$, and the x_{iu} (x_{iv}) which appear are replaced by their known values [3] in terms of the x_i. The right members are then expressed in terms of the X_i by means of (1), and when simplified with the use of (3) take the form

$$
(4)\quad
\begin{aligned}
36lX_{1u} &= 6BX_1 + (60C_0 - 72A)X_2 + (12C_4 - 45AB + 108)X_3\\
&\quad + (36C_2 - 108B - 54A^2)X_4,\\
36lX_{2u} &= 12X_1 + 2BX_2 + (12C_4 - 45AB + 216)X_4,\\
36lX_{3u} &= -36X_2 - 2BX_3 + (60C_0 - 108A)X_4,\\
36lX_{4u} &= 12X_3 - 6BX_4;\\
36mX_{1v} &= 6AX_1 + (12C_1 - 45AB + 108)X_2 + (60C_5 - 72B)X_3\\
&\quad + (36C_3 - 108A - 54B^2)X_4,\\
36mX_{2v} &= -2AX_2 - 36X_3 + (60C_5 - 108B)X_4,\\
36mX_{3v} &= 12X_1 + 2AX_3 + (12C_1 - 45AB + 216)X_4,\\
36mX_{4v} &= 12X_2 - 6AX_4.
\end{aligned}
$$

---

[3] Lane, *Projective Differential Geometry of Curves and Surfaces*, Chicago, 1932, p. 111.

By means of transformation (1) all the remaining results of this section are readily established.

The tangents of Darboux through 0 are given by

$$X_2^3 + X_3^3 = 0 = X_4;$$

the equation of a quadric of Darboux,

$$x_2x_3 - x_1x_4 + hx_4^2 = 0 \qquad (h = h(u, v)),$$

becomes

(5) $\qquad X_2X_3 - X_1X_4 + KX_4^2 = 0, \qquad K = (9/2\beta\gamma)(\theta_{uv} + 2h);$

the equations of a canonical line of the second kind,

$$x_1 - k\phi x_2 - k\psi x_3 = 0 = x_4,$$

have the form

$$X_1 - (B/2)(1 + 4k)X_2 - (A/2)(1 + 4k)X_3 = 0 = X_4,$$

and its polar line with respect to a quadric of Darboux joins 0 to the point

$$(0, (A/2)(1 + 4k), (B/2)(1 + 4k), 1).$$

If the corresponding equations of a plane in the above-mentioned coördinate systems are

$$\sum_1^4 \xi_i x_i = 0, \qquad \sum_1^4 \eta_i X_i = 0,$$

then the transformation of Čech in Fubini's system,

$$\begin{aligned} \rho x_1 &= \xi_2\xi_3\xi_4 - j(\gamma\xi_2^3 + \beta\xi_3^3), \qquad \rho x_2 = -\xi_2\xi_3^2, \\ \rho x_3 &= -\xi_2^2\xi_3, \qquad x_4 = 0 \qquad\qquad (j = \text{const.}), \end{aligned}$$

is expressed in Green's system by the equations

$$\begin{aligned} \sigma X_1 &= \eta_2\eta_3\eta_4 + 3j(\eta_2^3 + \eta_3^3), \qquad \sigma X_2 = -\eta_2\eta_3^2, \\ \sigma X_3 &= -\eta_2^2\eta_3, \qquad X_4 = 0. \end{aligned}$$

A curve on S which passes through 0 in a non-asymptotic direction and satisfies the differential equation

$$dv - \lambda du = 0 \qquad (\lambda = \lambda(u, v)),$$

has for its osculating plane at 0 the plane represented by

$$2\lambda(\lambda x_2 - x_3) + (\beta - \gamma\lambda^3 - \theta_u\lambda + \theta_v\lambda^2 + \lambda')x_4 = 0 \qquad (\lambda' = v'' = \lambda_u + \lambda\lambda_v).$$

Under transformation of coördinates (1) this equation becomes

$$(6) \qquad 6\mu(\mu X_2 - X_3) + (9\mu^3 - 9 + A\mu^2 - B\mu - 9\mu')X_4 = 0,$$

where μ, μ', are defined by

$$(7) \qquad \mu = \lambda l/m, \qquad \mu' = l(\mu_u + \lambda\mu_v) = l\mu_u + m\mu\mu_v.$$

The osculating plane at 0 of the pangeodesic which passes through this point in a non-Darboux direction $X_3/X_2 = \mu$ is expressed by

$$(8) \qquad 2\mu(\mu^3 + 1)(\mu X_2 - X_3) + (3\mu^6 - 3 + A\mu^2 - B\mu^4)X_4 = 0,$$

and the quadric of Moutard in the non-asymptotic direction μ at 0 has the equation

$$(9) \qquad \mu^3(X_2X_3 - X_1X_4) + \mu^2(2 - \mu^3)X_2X_4 + \mu(2\mu^3 - 1)X_3X_4 + (A\mu^2 + B\mu^4 - \mu^6 - 2\mu^3 - 1)X_4^2 = 0.$$

3. The curve of intersection of a surface and a quadric. The equation of the most general quadric having second-order contact with the surface S at the point 0 is

$$(10) \qquad XY - Z + K_2XZ + K_3YZ + K_4Z^2 = 0,$$

where K_2, K_3, K_4 are arbitrary functions of u and v. The curve of intersection C of this quadric and S is expressed by (10) and

$$(11) \qquad 0 = X^3 - K_2X^2Y - K_3XY^2 + Y^3 + (A - K_2^2)X^3Y - (2K_2K_3 + K_4)X^2Y^2 + (B - K_3^2)XY^3 + \cdots,$$

the latter equation being obtained by solving (10) for Z as a power series in X, Y and then subtracting from (2). The curve C has a triple-point at 0 with triple-point tangents satisfying the equations

$$(12) \qquad X^3 - K_2X^2Y - K_3XY^2 + Y^3 = 0 = Z.$$

Comparison of equations (10) and (12) leads to the following conclusion:

Two arbitrary non-asymptotic tangents to S at 0 determine a pencil of quadrics having second-order contact with S at 0 and meeting S in a one-parameter family of curves each containing these tangents as two of its triple-point tangents. The residual tangent at 0 is the same for all the curves of the family.

A particular quadric of such a pencil is uniquely determined, ordinarily, by characterizing the osculating plane at 0 of *any one* of the three branches of the curve C.[4]

[4] The Moutard pencils of quadrics are the only pencils for which this is not the case. But, as is shown in §§ 5, 6, for such a pencil of quadrics in a direction μ the

Several particular examples of the above theorem are worth noting. If two of the triple-point tangents lie in distinct Darboux directions at 0, the third tangent must lie in the residual Darboux direction and the resulting quadrics are the quadrics of Darboux.

If two of the tangents coincide in a non-asymptotic direction $X_3/X_2 = \mu$, the third must lie in the direction $-1/\mu^2$, and the quadrics determined form the Moutard pencil of quadrics in the direction μ. If $\mu^3 = -1$ the triple-point tangents all coincide in one of the Darboux directions, while if $\mu^3 = 1$ two of the tangents coincide in a Segre direction and the third tangent lies in the conjugate Darboux direction.

If two of the tangents lie in distinct Segre directions, the third is seen to lie in that Darboux direction which is conjugate to the residual Segre direction. Every Darboux direction thus determines a different pencil of such quadrics, so that each of the three pencils may be designated as the Darboux-Segre pencil of quadrics in the particular Darboux direction considered.

The Darboux quadrics and the three Darboux-Segre pencils of quadrics are special cases of the pencil

$$XY - Z - \frac{\mu^3+1}{\mu} XZ - \frac{\mu^3+1}{\mu^2} YZ + K_4 Z^2 = 0 \qquad (K_4 \text{ arbitrary}),$$

which is determined by the tangents in the directions $\mu\omega, \mu\omega^2, -1/\mu^2$, where ω is a primitive cube root of unity.

The quadrics of Darboux are thus seen to constitute the only pencil which can be determined by three tangents apolar to the asymptotic tangents at 0. Similarly, there exists a unique triple of tangents apolar to a given pair of conjugate tangents $X_3/X_2 = \pm \mu$, $X_4 = 0$, which according to the above theorem can determine a pencil of quadrics. These quadrics, whose equations can be shown to have the form

$$XY - Z - 3\mu^2 XZ - (3/\mu^2) YZ + K_4 Z^2 = 0, \qquad (K_4 \text{ arbitrary}),$$

were defined by Davis [5] in an entirely different manner.

4. Quadrics associated with the tangents of Darboux and of Segre. The theorem of the last section is applied here to the quadrics of Darboux, the Moutard pencils in the Segre directions, and the Darboux-Segre pencils of quadrics.

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osculating plane at 0 of a properly chosen branch of C does determine a unique quadric of the pencil, except when μ is a Darboux direction. In this latter case the osculating plane at 0 of each of the three branches of C coincides with the tangent plane to S, unless the quadric is the quadric of Moutard of the pencil.

[5] "Contributions to the theory of conjugate nets," *Chicago Dissertation* (1933), p. 12.

The surface S and an arbitrary quadric of Darboux with parameter K meet in a curve C whose three branches are representable, as is seen from (2) and (5), by the series

$$Y = -\epsilon_i X + \tfrac{1}{3}(A\epsilon_i^2 + B\epsilon_i + K)X^2 + mX^3 + \cdots,$$
$$Z = -\epsilon_i X^2 + \tfrac{1}{3}(A\epsilon_i^2 + B\epsilon_i + K)X^3 + \cdots, \qquad (i = 1, 2, 3),$$

where

$$-9\epsilon_i^2 m = 2AB\epsilon_i + 2B^2 + AK + 3BK\epsilon_i^2 + K^2\epsilon_i + 3\sum_{j=0}^{5} C_j(-\epsilon_i)^j,$$
$$(13) \qquad \epsilon_1 = 1, \ \epsilon_2 = \omega, \ \epsilon_3 = \omega^2$$

and ω is a primitive cube root of unity. The osculating plane at 0 of each branch of C has the equation

$$(14) \qquad \epsilon_i^2 X + \epsilon_i Y + \tfrac{1}{3}(A\epsilon_i^2 + B\epsilon_i + K)Z = 0,$$

from which it follows that *these osculating planes are coaxial if, and only if, the given quadric is the quadric of Wilczynski* ($K = 0$). *The common axis of the three planes then becomes the canonical line of the first kind for which* $k = -5/12$, *which we shall designate as the first axis of Bompiani.*[6]

It may be remarked here that the Moutard quadric in a Darboux direction $-\epsilon_i$ meets a Darboux quadric (5) in the asymptotic tangents through 0 and in a conic lying in the plane (14). As a consequence the Moutard quadrics in the three Darboux directions have a residual point of intersection on the first axis of Bompiani.

If the parameter K is left arbitrary, then the planes (14) correspond under a transformation of Čech with parameter j to three collinear points if, and only if, $K = -18j$ (the line thereby determined being the polar with respect to a Darboux quadric of the first axis of Bompiani, independently of the value of K). This equation defines a correspondence at each point of the surface between the Čech transformations and those Darboux quadrics for which K is independent of u, v. Thus, the quadrics of Lie and Fubini for which $K = -9/2$ and -3 respectively, characterize the transformations of Čech for which $j = 1/4$ and $1/6$, whereas the transformation of Segre ($j = 1$) characterizes the quadric of Darboux for which $K = -18$.

The osculating plane (14) is a stationary osculating plane if, and only if,

$$-9\epsilon_i m = (A\epsilon_i^2 + B\epsilon_i + K)^2;$$

the value of K is thereby uniquely determined[7] and when substituted into

[6] This line was first characterized in a more complicated way by Bompiani in the *Rendiconti dei Lincei*, ser. 5, vol. 33_1 (1924), pp. 85-90.

[7] The value of K depends on the particular Darboux direction chosen, so that there exist three distinct Darboux quadrics with the property considered.

(14) gives as the equation of the stationary osculating plane

$$(A\epsilon_i - B)(\epsilon_i X + Y) + [\sum_{j=0}^{5} C_j(-\epsilon_i)^j]Z = 0. \tag{15}$$

The Moutard pencil of quadrics at 0 in the Segre direction $X_3/X_2 = \epsilon_i$ has the form

$$XY - Z + \epsilon_i^2 XZ + \epsilon_i YZ + K_i Z^2 = 0, \tag{16}$$

with K_i as parameter of the pencil. The curve of intersection of the surface S and an arbitrary quadric of this pencil has a branch in the conjugate Darboux direction, the osculating plane at 0 of this branch being expressed by

$$\epsilon_i^2 X + \epsilon_i Y + \tfrac{1}{4}(A\epsilon_i^2 + B\epsilon + K_i)Z = 0. \tag{17}$$

Since the equation of the osculating plane at 0 of the curve of Darboux in the direction $-\epsilon_i$ is

$$\epsilon_i^2 X + \epsilon_i Y + \tfrac{1}{6}(A\epsilon_i^2 + B\epsilon_i - 18)Z = 0, \tag{18}$$

it follows that the planes (17) and (18) coincide only if

$$K_i = -(A/3)\epsilon_i^2 - (B/3)\epsilon_i - 12.$$

This value of K_i determines a unique quadric of the pencil (16), and since there are three Moutard pencils in the Segre directions at 0, three corresponding quadrics are obtained in all. *These quadrics intersect in the asymptotic tangents through* 0 *and in a residual point lying on the canonical line of the first kind for which* $k = -1/12$. *Under the transformation of Čech with* $j = 1/2$ *the three planes* (18) *are carried into collinear points on the second axis of Čech.*

The condition that the osculating plane (17) contain a line l joining 0 to an arbitrary point $(0, a, b, 1)$ requires that

$$K_i = -(4a + A)\epsilon_i^2 - (4b + B)\epsilon_i.$$

The line l obtains in this way a single quadric out of each of the three Moutard pencils in the Segre directions, the residual point of intersection P of these quadrics having the coördinates

$$(4a + A)(4b + B), \ 4a + A, \ 4b + B, \ 1.$$

As the line l *varies in the bundle at* 0, *the point* P *generates the quadric of Wilczynski*; P *lies on* l *if, and only if,* l *is the first axis of Bompiani.*

The quadric of Moutard in the Segre direction ϵ_i at 0 is that quadric of the pencil (16) for which

$$K_i = A\epsilon_i^2 + B\epsilon_i - 4;$$

the osculating plane (17) determined by this value of K_i can also be described as the plane containing the non-degenerate conic of intersection of the Moutard quadrics in the conjugate directions $\pm\epsilon_i$. The residual point of intersection of the Moutard quadrics in the three Segre directions lies on the canonical line of the first kind for which $k = -3/4$, while the transformation of Čech for which $j = 1/6$ carries the three planes (17) determined by these quadrics into collinear points on the second directrix of Wilczynski.

The Darboux-Segre pencils of quadrics will now be considered. For each choice of ϵ_i let ϵ_i, ϵ_j, ϵ_k represent the three values in (13). Then the three directions

$$X_3/X_2 = -\epsilon_i, \qquad X_3/X_2 = \epsilon_j, \qquad X_3/X_2 = \epsilon_k$$

at 0 determine a Darboux-Segre pencil of quadrics in the Darboux direction $-\epsilon_i$, having the equation

$$(19) \qquad XY - Z - 2\epsilon_i^2XZ - 2\epsilon_iYZ + K_iZ^2 = 0 \qquad (K_i \text{ arbitrary}).$$

If the osculating planes at 0 of the three branches of the curve of intersection of the surface S and one of these quadrics be denoted by π_{ii}, π_{ij}, π_{ik} corresponding to the respective directions $-\epsilon_i$, ϵ_j, ϵ_k, then

$$\pi_{ii}: \quad \epsilon_i^2X + \epsilon_iY + (A\epsilon_i^2 + B\epsilon_i + K_i)Z = 0,$$

$$\pi_{ih}: \quad \epsilon_h^2X - \epsilon_hY + P_{ih}Z = 0 \qquad (h = j, k),$$

where

$$P_{ih} = \frac{-A\epsilon_h - B + 4\epsilon_h^2 + K_i\epsilon_h^2}{2\epsilon_i^2 + 4\epsilon_i\epsilon_h + 3\epsilon_h^2}.$$

The condition that the plane π_{ii} coincide with the osculating plane of a curve of Darboux at 0 determines a unique quadric out of each Darboux-Segre pencil. *The residual point of intersection of these three quadrics is the point of intersection* (*besides* 0) *of the quadric of Fubini with the canonical line of the first kind for which* $k = -11/24$.

A unique quadric in each Darboux-Segre pencil is also obtained from the condition that the three planes π_{ii}, π_{ij}, π_{ik} be coaxial, the parameters K_i then having the values

$$K_i = -\tfrac{4}{5}(A\epsilon_i^2 + B\epsilon_i + 2) \qquad (i = 1, 2, 3).$$

The three quadrics thus found have their residual point of intersection on the canonical line of the first kind for which $k = -9/20$.

The locus of the line of intersections of the planes π_{ij}, π_{ik} for all quadrics

in the pencil (19) is a plane passing through the Segre tangent $X_3/X_2 = \epsilon_i$, $X_4 = 0$ and through the canonical line of the first kind for which $k = -3/4$. The quadric determined by the condition that the planes π_{ij}, π_{ik} contain this canonical line corresponds to the parametric value

$$K_i = -2A\epsilon_i{}^2 - 2B\epsilon_i - 4,$$

and the three quadrics so obtained for $i = 1, 2, 3$ have their residual point of intersection on this same line.

5. The Moutard pencils of quadrics. This section is concerned essentially with an examination of the singularity of the curve of intersection of the surface S and a quadric of the Moutard pencil. The curve determined by the Moutard quadric of the pencil is found to differ radically from the curves determined by the other quadrics of the pencil.

The Moutard pencil of quadrics determined at a point 0 of S by a direction $X_3/X_2 = \mu$ has the form

$$XY - Z + (1/\mu)(2 - \mu^3)XZ + (1/\mu^2)(2\mu^3 - 1)YZ + K_\mu Z^2 = 0, \tag{20}$$

where K_μ is an arbitrary function of u, v. The quadric of Moutard of the pencil is obtained when

$$\mu^3 K_\mu = A\mu^2 + B\mu^4 - (\mu^3 + 1)^2. \tag{21}$$

Let the equation

$$Z = n(Y - \mu X) \qquad (n \neq 0) \tag{22}$$

represent an arbitrary plane through the tangent $X_3/X_2 = \mu$, $X_4 = 0$, cutting the surface S in a curve s and a quadric of the pencil (20) in a conic q. Then s and q have third-order contact at 0 for all values of n and K_μ. Conversely, if every plane of the form (22) meets the surface S and a quadric Q in a pair of curves which have contact of the third order at 0, then Q belongs to the pencil (20). The quadric of Moutard (9) is the unique quadric of the pencil for which the contact between s and q is of the fourth order for all values of n.

If μ is not a Darboux direction, there are two planes of sextactic section [8] passing through the tangent $X_3/X_2 = \mu$, $X_4 = 0$, for which the contact between s and q is of the fifth order at 0. The values of n in (22) which determine these planes are the roots of the equation

$$hn^2 + (3 - 3\mu^6 - A\mu^2 + B\mu^4)\mu n + (\mu^3 + 1)\mu^2 = 0, \tag{23}$$

[8] Darboux, *Bulletin des Sciences Mathematiques et Astronomiques*, ser. 2, vol. 4 (1880), p. 366.

where h is defined by

$$h = 2(\mu^3 + 1)^3 - 3\mu^2(\mu^3 + 1)(A + B\mu^2) + \mu^2\alpha,$$

and

$$\alpha = \sum_0^5 C_i\mu^i. \tag{24}$$

The curve of intersection of S and an arbitrary quadric of the pencil (20) is given by equations (20) and (11). If μ is not a Darboux direction, this curve has two branches at 0 in the direction μ, whose equations are

$$Y = \mu X \pm rX^{3/2} + \cdots, \qquad Z = XY + \cdots, \tag{25}$$

where r satisfies the equation

$$r^2 = -\mu(A\mu^2 + B\mu^4 - \mu^6 - 2\mu^3 - 1 - \mu^3K_\mu)/(\mu^3 + 1).$$

If $r = 0$ the curve (25) is found to have a tacnode in the direction μ, so that in view of (21) *a necessary and sufficient condition that a quadric of the Moutard pencil in a non-Darboux direction* μ *at* 0 *be the quadric of Moutard* (9) *is that its curve of intersection with* S *have a tacnode at* 0 *in the direction* μ *instead of a cusp.*

For the case of the Moutard quadric (9) the terms of fifth degree in the series (11) can be shown to be

$$\sum_0^5 T_iX^{5-i}Y^i$$

where

$$T_0 = C_0, \qquad \mu^3T_1 = \mu^3C_1 + \mu^9 - 6\mu^6 + 12\mu^3 - 8,$$
$$\mu^4T_2 = \mu^4C_2 - 9\mu^9 + 3B\mu^7 + 27\mu^6 + 3A\mu^5 - 6B\mu^4 - 27\mu^3 - 6A\mu^2 + 18,$$
$$\mu^5T_3 = \mu^5C_3 + 18\mu^9 - 6B\mu^7 - 27\mu^6 - 6A\mu^5 + 3B\mu^4 + 27\mu^3 + 3A\mu^2 - 9,$$
$$\mu^6T_4 = \mu^6C_4 - 8\mu^9 + 12\mu^6 - 6\mu^3 + 1, \qquad T_5 = C_5.$$

The two tacnode branches of the curve of intersection of the surface S and the Moutard quadric (9) are now found to have the equations

$$Y = \mu X + \sigma_iX^2 + \cdots, \qquad Z = XY + \cdots \qquad (i = 1, 2), \tag{26}$$

where σ_1, σ_2 are the roots of the equation

$$(\mu^3 + 1)\sigma^2 + (3 - 3\mu^6 - A\mu^2 + B\mu^4)\sigma + h = 0. \tag{27}$$

If $r \neq 0$, the osculating plane at 0 of each cusp-branch (25) coincides with the tangent plane $Z = 0$, whereas if $r = 0$ the osculating planes at 0 are represented by the equations

$$Z = (\mu/\sigma_i)(Y - \mu X) \qquad (i = 1, 2). \tag{28}$$

Setting $\mu/\sigma_i = n_i$ and using (27) we find that n_1, n_2 are the roots of equation (23). Hence, *the osculating planes at* 0 *of the tacnode branches of the curve of intersection of the surface S and the Moutard quadric in the non-Darboux direction μ at* 0 *coincide with the planes of sextactic section which pass through the tangent line in the direction μ.*

Comparison of the discriminants of (23) and (27) shows that the two planes of sextactic section through the line $X_3/X_2 = \mu$, $X_4 = 0$ coincide if, and only if, the tacnode branches (26) have maximum-order contact at 0, this situation occurring for twelve values [9] of μ.

The results concerning the Moutard pencil of quadrics in a Darboux direction $X_3/X_2 = -\epsilon_i$ at 0 will be stated briefly. Each quadric Q of the pencil meets the surface S in a curve C whose triple-point tangents all coincide in the given Darboux direction. If Q is not the Moutard quadric of the pencil, this curve has a superlinear branch of order 3 and the osculating plane at 0 of each branch coincides with the plane $Z = 0$. If Q is the Moutard quadric then the curve C consists of two cusped branches and one linear branch. The osculating planes of the cusped branches at 0 coincide with the tangent plane to S, while the osculating plane of the linear branch at 0 is none other than the plane (15). This plane can also be obtained as a limiting case of (28) as μ approaches $-\epsilon_i$.

6. Cones associated with the Moutard pencils of quadrics. The problem of characterizing individual members of the Moutard pencils of quadrics in the Segre directions at a point of a surface has already been discussed. We now consider this problem for an arbitrary direction.

The curve of intersection of the surface S and an arbitrary quadric of the Moutard pencil (20) in a non-Darboux direction μ at 0 has a linear branch in the direction $-1/\mu^2$. The equations of this branch are

$$Y = -\frac{1}{\mu^2}X + \frac{A\mu^8 + B\mu^4 - (\mu^6 - 1)^2 + \mu^6 K_\mu}{\mu^6(\mu^3 + 1)^2}X^2 + \cdots, \qquad Z = XY + \cdots,$$

so that its osculating plane at 0 is of the form

$$\mu^2(\mu^3 + 1)(X + \mu^2 Y) + [A\mu^8 + B\mu^4 - (\mu^6 - 1)^2 + \mu^6 K_\mu]Z = 0. \tag{29}$$

The condition that this plane contain a line l joining 0 to an arbitrary

[9] Wilczynski, "Fifth memoir," *Transactions of the American Mathematical Society*, vol. 10 (1909), p. 293.

point $(0, a, b, 1)$ determines a unique quadric Q_μ of the pencil, since the parameter K_μ must satisfy the equation

$$(30) \quad \mu^6 K_\mu = (\mu^6 - 1)^2 - A\mu^8 - a(\mu^3+1)^2\mu^2 - B\mu^4 - b(\mu^3+1)^2\mu^4.$$

Associated with the direction μ are two others, $\mu\omega$, $\mu\omega^2$ which form with μ a triple of directions apolar to the asymptotic directions. Thus two other quadrics, $Q_{\mu\omega}$, $Q_{\mu\omega^2}$, belonging to the Moutard pencils at 0 in the directions $\mu\omega$, $\mu\omega^2$, can be defined by this line l in exactly the same way as is Q_μ, the equations of these quadrics being obtained from (20) and (30) by replacing μ by $\mu\omega$, $\mu\omega^2$, respectively. The three quadrics Q_μ, $Q_{\mu\omega}$, $Q_{\mu\omega^2}$ intersect in the asymptotic tangents through 0 and in a residual point whose join with 0 is a line l' containing the point

$$(31) \quad \begin{aligned} &X_1 = 0, \qquad X_2 = (2\mu^3 - 1)[A\mu^6 + a(\mu^3+1)^2], \\ &X_3 = \mu^3(2-\mu^3)[B + b(\mu^3+1)^2], \qquad X_4 = \mu^3(2-\mu^3)(2\mu^3-1). \end{aligned}$$

The lines l and l' coincide if, and only if,

$$(32) \qquad a = -A\mu^6/(2\mu^6+1), \qquad b = -B/(\mu^6+2),$$

so that the line l and the quadric Q_μ are uniquely determined by the given direction μ. As μ varies, l generates the quadric cone given by

$$(33) \qquad 3XY + 2AYZ + 2BXZ + ABZ^2 = 0.$$

If, however, the line l is kept fixed and the direction μ is varied, then l' generates a cone which in general is of the fourth order but for special positions of l may be of the second order. To determine the conditions under which this occurs, we note that any quadric cone generated by the line l' must be of the form

$$(34) \qquad XY + pYZ + qXZ + rZ^2 = 0,$$

where p, q, r are functions of u, v. Substitution of equations (31) into (34) leads to seven equations of which the following five are independent:

$$\begin{aligned} &b(p - a - A) = 0, \qquad a(q - b - B) = 0, \\ &Ab - 3ab - 5pb - 4q(a + A) + 4r = 0, \\ &aB - 3ab - 5aq - 4p(b + B) + 4r = 0, \\ &5AB + Ab + aB + 14ab + p(11b - 9B) + (11a - 9A)q + 33r = 0. \end{aligned}$$

These equations have five sets of solutions for a, b, p, q, r which are exhibited in the following table:

Case	a	b	p	q	r
(I)	$-A/3$	$-B/3$	$2A/3$	$2B/3$	$AB/3$
(II)	0	0	$-A/3$	$-B/3$	$-AB/3$
(III)	$-4A/9$	$-4B/9$	$5A/9$	$5B/9$	$7AB/27$
(IV)	0	$-4B/9$	A	B	$5AB/9$
(V)	$4A/9$	0	A	B	$5AB/9$

Hence, *a necessary and sufficient condition that the line l' generate a quadric cone as μ varies is that the fixed line l, which is the axis of the pencil of planes* (29), *join* 0 *to the point* $(0, a, b, 1)$ *where a, b have any one of the five pairs of values listed in the table.*

Five quadrics belonging to the Moutard pencil in an arbitrary non-Darboux direction [10] μ at 0 are thus characterized, each quadric having associated with it a quadric cone and a fixed line l. In cases (I) and (II) of the table this line is the first axis of Bompiani and the first edge of Green respectively, while in case (III) it is the canonical line of the first kind for which $k = -17/36$. The lines determined in cases (IV) and (V) can be described as the polars with respect to a quadric of Darboux of the lines $A_X B_Y$, $A_Y B_X$, where A_X, A_Y and B_X, B_Y are the intersections of the asymptotic tangents through 0 with the canonical lines of the second kind for which $k = -17/36, -1/4$.

The quadric cones associated with these five quadrics are all mutually tangent along the first axis of Bompiani, the common tangent plane passing through the second canonical tangent. Consideration of these cones readily leads to the characterization of a number of new canonical lines of the first kind.

The cone (33), which is also one of the cones listed in the table, can be obtained in still another way. The line joining 0 to the residual point of intersection of the three quadrics of Moutard in the directions μ, $\mu\omega$, $\mu\omega^2$ generates the cone (33) as μ varies.[11]

There is a certain cone of class 3 closely associated with the Moutard quadrics at 0. The quadrics of Moutard in the conjugate directions μ, $-\mu$ meet in the asymptotic tangents through 0 and in a conic whose plane has the equation

$$2\mu^2(X + \mu^2 Y) + (A\mu^2 + B\mu^4 - \mu^6 - 1)Z = 0.$$

As μ varies this plane envelopes the cone, which in local plane coördinates $\eta_1, \cdots, \eta_4$ has the equations

[10] If μ is a Darboux direction these quadrics, as well as the quadric determined by (32), all coincide with the Moutard quadric.

[11] Bompiani, "Contributi alla geometria proiettivo-differenziale di una superficie," *Bollettino della Matematica Italiana*, vol. 3 (1924), pp. 97-100.

$$\eta_2{}^3+\eta_3{}^3-\eta_2\eta_3(A\eta_2+B\eta_3-2\eta_4)=0=\eta_1.$$

The cusp-planes of this cone pass through the Segre tangents at 0 and intersect in the first directrix of Wilczynski.

7. Envelopes of Moutard quadrics. The problem of determining the envelope of a family of Moutard quadrics is of sufficient importance to warrant consideration. This problem has been discussed only by Kimpara,[12] to the best of the writer's knowledge, and then merely for a special case.

The one-parameter family of Moutard quadrics at a point 0 of a surface S has an enveloping surface Σ which contains the asymptotic tangents through 0 as a locus of singular points. Elimination of μ between (9) and its μ-derivative shows that Σ is an algebraic surface of order 14. The characteristic curve determined by Σ and a particular quadric of the family in the direction μ is a conic lying in the osculating plane at 0 of the pangeodesic which passes through 0 in the direction μ. Hence, *the lines joining* 0 *to the points on the edge of regression of* Σ *generate the cone of Segre.*

We now proceed with the problem of determining the envelope of a two-parameter family of Moutard quadrics. Let $C^{(1)}$ be a one-parameter family of curves on S which satisfy the differential equation

$$dv-\lambda du=0 \qquad (\lambda=\lambda(u,v)), \tag{35}$$

and let it be assumed that these curves simply cover the portion of the surface under consideration and are nowhere tangent to an asymptotic curve. The tangents to $C^{(1)}$ then determine a two-parameter family of Moutard quadrics $Q^{(2)}$, one at each point of S. A particular point 0 therefore has associated with it a curve C_μ of the family $C^{(1)}$, whose tangent at 0 is

$$X_3/X_2=\mu, \quad X_4=0 \qquad (\mu=\lambda l/m),$$

and a quadric Q_μ of the family $Q^{(2)}$. An arbitrary curve C_ν on S, which passes through 0 in a direction $X_3/X_2=\nu$, determines at each of its points a unique Moutard quadric belonging to $Q^{(2)}$, the quadrics so obtained forming a one parameter family $Q^{(1)}$ with an envelope E_ν. The quadric Q_μ at 0 is a member of this family and hence touches E_ν along a characteristic curve Γ_ν.

These curves Γ_ν and the envelope E of the family of quadrics $Q^{(2)}$ depend on the given curves $C^{(1)}$, and their properties differ widely in two separate cases, namely, when the curves $C^{(1)}$ are curves of Darboux and when they are nowhere tangent to a curve of Darboux.

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[12] "Sur l'enveloppe des quadriques de Moutard," *Ryojun College of Engineering, Commemoration volume* (1934), pp. 33-39.

(i) Suppose first that the curves $C^{(1)}$ form a one-parameter family of curves of Darboux. The quadric Q_μ is the Moutard quadric in the Darboux direction $X_3/X_2 = \mu$ at 0, and has the equation

$$(36) \qquad G \equiv X_2X_3 - X_1X_4 - 3\mu^2 X_2X_4 + 3\mu X_3X_4 + (B\mu - A\mu^2)X_4^2 = 0.$$

The characteristic curve Γ_ν determined by Q_μ and a curve C_ν through 0 is therefore given by (36) and

$$G_u + G_v(dv/du) = 0,$$

where dv/du refers to the direction of the tangent to C_ν at 0, and since $dv/du = \nu m/l$ this may be written

$$(37) \qquad lG_u + \nu m G_v = 0.$$

The derivatives of A and B which appear in this equation may be readily evaluated. For,

$$A = -(3/2)m\psi, \qquad A_u = -(3/2)(m_u\psi + m\psi_u),$$
$$m = 1/(\beta\gamma^2)^{1/3}, \qquad m_u = -(m/3)(\beta_u/\beta) - (2m/3)(\gamma_u/\gamma),$$

and consequently,

$$18lA_u = 4AB - 27n\psi_u.$$

Now from (3) it is seen that $n\psi_u$ can be expressed in terms of the coefficients appearing in series (2), and hence lA_u is expressible in like fashion. Thus we find

$$36lA_u = 278AB - 648 - 48C_1 - 24C_4,$$
$$36mB_v = 278AB - 648 - 48C_4 - 24C_1,$$
$$36mA_v = 52A^2 + 288B - 24C_2 - 120C_5,$$
$$36lB_u = 52B^2 + 288A - 24C_3 - 120C_0.$$

With the aid of formulas (4) the left member of (37) can now be expressed in terms of μ, ν, the coefficients of series (2), and the local coördinates $X_1, \cdots, X_4$. Elimination of the term X_1X_4 from the resulting equation by means of (36) finally yields the simplified result

$$(38) \qquad 36(\mu - \nu)(\mu X_2 - X_3) + LX_2X_4 + MX_3X_4 + NX_4^2 = 0,$$

where the coefficients L, M, N have the values

$$L = 12B\mu(\mu + 2\nu) - 36A + 216(\nu - \mu),$$
$$M\mu = 12\mu(A\mu + B)(\mu - \nu) - L,$$
$$N \doteq 72A(2\nu - \mu) - 72B\mu^2(2\mu - \nu) + B^2(54\nu + 40\mu) - A^2\mu^2(54\mu + 40\nu)$$
$$+ 131AB\mu(\nu - \mu) + 12\sum_0^5 C_i\mu^4[(5 - i)\mu + i\nu].$$

If $\nu \neq \mu$, the equations of Γ_ν can also be written in the non-homogeneous form

$$Y = \mu X \pm \tfrac{1}{6}\left(\frac{L\mu + M\mu^2}{\nu - \mu}\right)^{1/2} X^{3/2} + \cdots, \qquad Z = XY + \cdots \qquad (\nu \neq \mu),$$

which simplify to

$$Y = \mu X \pm [\tfrac{1}{3}(A - B\mu^2)]^{1/2} X^{3/2} + \cdots, \qquad Z = XY + \cdots \qquad (\nu \neq \mu).$$

Hence, if $\nu \neq \mu$, the characteristic curve Γ_ν is a quartic curve with a cusp at 0 and a cusp-tangent in the Darboux direction μ.

If C_ν is tangent at 0 to the curve of Darboux C_μ, then the characteristic curve Γ_ν consists of the asymptotic tangents through 0 and of a non-degenerate conic lying in the plane whose equation is

$$(39) \qquad 18(A\mu + B)(\mu X_2 - X_3) + [30\alpha - 36\mu^2(A\mu + B) - 47(A^2\mu^2 - B^2)]X_4 = 0,$$

α being defined in (24). Since this conic is tangent to C_μ at 0, it follows that if C_ν and C_μ coincide, then C_μ is part of the edge of regression of the envelope E_ν.

The points where the Quadric Q_μ touches the envelope E of the two-parameter family of Moutard quadrics $Q^{(2)}$ are determined by the simultaneous solutions of equations (36), (38) and (39), where in (38) the value of ν is arbitrary but different from μ. Since the plane (39) intersects the cone (38) in the Darboux tangent $X_3/X_2 = \mu$, $X_4 = 0$ and in another line l which passes through 0, and since these two lines meet the quadric Q_μ at 0 and at some other point P not lying on S, it therefore follows that *the envelope E consists of the surface S and of another sheet generated by the point P.*

(ii) We now assume that the curves $C^{(1)}$ are nowhere tangent to a curve of Darboux. If $H = 0$ is the equation (9) of the Moutard quadric Q_μ in the direction μ at 0, then the characteristic curve Γ_ν determined by Q_μ and a curve C_ν through 0 has the form

$$H = 0, \qquad lH_u + \nu m H_v = 0.$$

The second of these equations can be expressed in terms of μ, ν, the coefficients of series (2), and the local coördinates $X_1, \cdots, X_4$, and when combined with (9) to eliminate the term X_1X_4 becomes

$$(40) \qquad 12\mu^4(\mu X_2 - X_3)[(2\nu - 3\mu - \mu^3\nu)X_2 - (2\mu^3 - 3\mu^3\nu - 1)X_3] + DX_2X_4 + EX_3X_4 + FX_4^2 = 0,$$

where D, E, F have the values

$$
\begin{aligned}
D = {} & 4A\mu^5(2\mu^3\nu - 9\mu + 2\nu) + 4B\mu^5(\mu^3 + 6\mu^2\nu - 2) \\
& + 12\mu^3[\mu^6(\mu - 4\nu) - 10\mu^3(\mu - \nu) + 7\mu - 4\nu] \\
& - 72\mu^4(\mu^3 + 1)(l\mu_u + \nu m\mu_v), \\
E\mu = {} & 12\mu^3(\nu - \mu)(3\mu^6 - 3 + A\mu^2 - B\mu^4) - D, \\
F = {} & 12A\mu^4[(5\mu^3 + 11) + \mu^2\nu(\mu^3 + 13)] \\
& + 12B\mu^3[(13\mu^3 + 1) + \mu^2\nu(11\mu^3 + 5)] \\
& + A^2\mu^5(54\mu + 40\nu) + B^2\mu^6(40\mu + 54\nu) \\
& + AB\mu^4[\mu(45\mu^3 + 176) + \nu(176\mu^3 + 45)] \\
& + 12\mu[\mu(\mu^9 - 18\mu^6 - 21\mu^3 - 2) + \nu(-2\mu^9 - 21\mu^6 - 18\mu^3 + 1)] \\
& - 12\mu^3 \sum_0^5 C_i\mu^i[(5 - i)\mu + i\nu] \\
& + 36\mu^2(3 - 3\mu^6 - A\mu^2 + B\mu^4)(l\mu_u + \nu m\mu_v).
\end{aligned}
$$

The characteristic curve Γ_ν is therefore a quartic curve with a double point at 0, one nodal tangent always lying in the given direction μ, the other depending on ν. This second tangent lies in the direction μ if, and only if, $\nu = \mu$, and lies in the direction ν only if $\nu = \mu$ or $\nu = -1/\mu^2$. The osculating plane at 0 of that branch of Γ_ν which lies in the direction μ has the equation

$$24\mu^4(\mu^3 + 1)(\mu - \nu)(\mu X_2 - X_3) - (D + E\mu)X_4 = 0,$$

which reduces, when $\nu \neq \mu$, to (8). Hence, if $\nu \neq \mu$, the osculating plane at 0 of that nodal branch of the characteristic curve Γ_ν which lies in the direction μ is independent of ν and is the osculating plane of the pangeodesic in the direction μ.

If $\nu \neq \mu$, the osculating plane of the second nodal branch of Γ_ν at 0 is expressible in the form

$$
\begin{aligned}
24\mu^4(\mu^3 + 1)(\mu - \nu)[(2\nu - 3\mu - \mu^3\nu)X_2 - (2\mu^3 - 3\mu^2\nu - 1)X_3] \\
+ [D(2\mu^3 - 3\mu^2\nu - 1) + E(2\nu - 3\mu - \mu^3\nu)]X_4 = 0.
\end{aligned}
$$

As ν varies this plane generates an axial pencil, its limiting position as ν approaches μ being given by

$$
\begin{aligned}
(41)\quad & 6\mu(\mu^3 + 1)(\mu X_2 - X_3) \\
& + [A\mu^2(11 - 4\mu^3) + B\mu(4 - 11\mu^3) + 9\mu^6 - 9 + 36(\mu^3 + 1)\mu']X_4 = 0,
\end{aligned}
$$

where μ' is defined in (7). The plane (41) depends on the curve C_μ alone, being independent of the other curves of the family $C^{(1)}$, and together with the osculating plane of C_μ at 0, the osculating plane of the pangeodesic in the direction μ at 0, and the tangent plane $X_4 = 0$, determines an invariant cross-

ratio of -4. Hence, the plane (41) coincides at every point of the surface S with the osculating plane of the curve C_μ if, and only if, the curves $C^{(1)}$ are pangeodesics.

If $\nu=\mu$, (40) reduces to

$$[X_4-n_1(X_3-\mu X_2)][X_4-n_2(X_3-\mu X_2)]=0, \tag{42}$$

where n_1, n_2 are the roots of the equation

$$Fn^2+En-12\mu^4(\mu^3+1)=0 \qquad (\nu=\mu). \tag{43}$$

The characteristic curve Γ_ν therefore consists of two non-degenerate conics determined by the intersections of the quadric Q_μ and the planes (42). The harmonic conjugate of the plane $X_4=0$ with respect to these two planes has the form

$$24\mu^4(\mu^3+1)(\mu X_2-X_3)+EX_4=0 \qquad (\nu=\mu),$$

from which it follows that this plane, the osculating plane of C_μ at 0, the osculating plane of the pangeodesic in the direction μ at 0, and the plane $X_4=0$, determine a cross-ratio of -2.

The curves $C^{(1)}$ cannot be chosen so that at each point of S the two planes (42) coincide with the planes of sextactic section (28) which pass through the tangents to $C^{(1)}$. However, comparison of equations (43) and (23) shows that this situation does occur at a particular point 0 of S if, and only if, C_μ has second-order contact at 0 with any one of a set of twelve pangeodesics.

The quadric Q_μ touches the envelope E of the two-parameter family of Moutard quadrics $Q^{(2)}$ at the points determined by equations (9), (40) and (42), where the value of ν in (40) is arbitrary but different from μ. Since the two planes (42) meet the cone (40) in two lines l_1, l_2 which pass through 0 and are ordinarily distinct, and in the tangent $X_3/X_2=\mu$, $X_4=0$, and since these three lines intersect Q_μ at 0 and at two other points P_1, P_2, it follows that *the envelope E consists of the surface S and of two other sheets, those generated by the points P_1, P_2.*

The lines l_1, l_2 coincide if, and only if,

$$E^2+48F\mu^4(\mu^3+1)=0, \tag{44}$$

where $\nu=\mu$ in the expressions for E, F. Hence, at a particular point 0 of S there are two planes either of which may serve as the osculating plane of the curve C_μ in order that l_1 and l_2 coincide. It is impossible, however, to choose the curves $C^{(1)}$ so that the lines l_1, l_2 coincide for every point of S, since equation (35) which determines the curves $C^{(1)}$ cannot satisfy (44) which is now a differential equation of the second order and second degree in dv/du.

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A GENERALIZATION OF LOCAL CLASS FIELD THEORY.*

By O. F. G. Schilling.[1]

In this paper we present a new generalization of local class field theory. The underlying ground fields will be perfect with respect to an archimedian valuation whose associated value group contains the additive group of all rational numbers and a finite number of infinite cycles which are generated by irrational real numbers. The resulting theory has many similarities with the theory of fields of formal power series of several variables.[2] We put stress on the investigation of the class group of algebras; it turns out that there exist in general division algebras which do not possess unramified splitting fields. Consequently, the theory has an entirely different aspect compared with the local class field theory as developed in earlier papers of M. Moriya and the author.[3] Only in two very special cases one can reestablish the classical results. We incorporated in the paper a result concerning the relationship between finite separable extensions and normal algebras over everywhere dense subfields.[4] Moreover, we prove the existence of perfect fields for which our general theory was developed. These and other examples seem to indicate that an axiomatic treatment of the algebraic structure of general perfect fields along the same lines as in the case of discrete perfect fields [5] will be quite involved, in particular the structure of the fields considered as additive groups is very complicated.

1. Preparations. Let k be an abstract field on which an archimedian valuation $\mathfrak{p}$ is defined, i. e. k is a field whose elements $a \neq 0$ can be mapped upon elements $v(a) = \alpha$ of a group $\Gamma(k)$ of real numbers; moreover, the function $v(a)$ satisfies the following conditions

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[1] Johnston Scholar at the Johns Hopkins University for 1937-1938.

[2] O. F. G. Schilling, "Arithmetic in fields of formal power series in several variables," *Annals of Mathematics*, vol. 38 (1937).

[3] M. Moriya and O. F. G. Schilling, (1) "Zur Klassenkörpertheorie über unendlichen perfekten Körpern," *Journal of the Faculty of Science Hokkaido Imperial University Sapporo* (Japan), ser. I, vol. 5 (1936); (2) "Divisionsalgebren über unendlichen perfekten Körpern," *loc. cit.*, ser. I, vol. 6 (1937).

[4] O. F. G. Schilling, "A note on infinite perfect fields," *Abstract in the Bulletin of the American Mathematical Society*, vol. 44 (1938), no. 1, part 1.

[5] O. F. G. Schilling, "The structure of local class field theory," *American Journal of Mathematics*, vol. 60 (1938).

$$v(ab) = v(a) + v(b)$$
$$v(a+b) \geqq \min(v(a), v(b)).$$

In order to include the element 0 of k we define $v(0) = \infty$ and assume the usual formal relations to hold between the elements of $\Gamma(k)$ and the symbol ∞.

All elements a in k whose values are non-negative form the *maximal order* $\mathfrak{o}$ of all $\mathfrak{p}$-adic integers in k. The maximal order $\mathfrak{o}$ contains a *prime ideal* $\mathfrak{p}$ which is given by all elements of k whose values are greater than the zero-element of $\Gamma(k)$. The ring of residual classes $\mathfrak{o}/\mathfrak{p}$ is a field $\boldsymbol{k}$, the so-called *residue class field* of k with respect to the valuation $\mathfrak{p}$.[6]

We shall assume throughout the paper that the fields k and $\boldsymbol{k}$ have the same characteristic χ. However, most of the theorems we shall prove will also hold if we drop this assumption.

All elements of k form the points of a metric space if we define the distance $\delta(a, b)$ between two elements a and b of k to be equal to $C^{-\delta(a-b)}$ where C denotes any positive number greater than 1. The space k need not be complete with respect to the metric, but in any case we can complete it to the so-called *perfect $\mathfrak{p}$-adic closure* by adjoining all fundamental sequences which are defined by the metric.

It turns out that by completing the field k we enlarge neither the value group $\Gamma(k)$ nor the field of residual classes $\boldsymbol{k}$.

The essential feature of perfect $\mathfrak{p}$-adic fields k is expressed by Hensel's criterion of reducibility:[7]

If the polynomial $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ with integral coefficients in the perfect field k decomposes mod $\mathfrak{p}$ into the product $\prod_{i=1}^{m} \phi_i(x)$ of relatively prime polynomials $\phi_i(x)$ with coefficients in $\boldsymbol{k}$ then $f(x)$ already decomposes in k and $f(x) = \prod_{i=1}^{m} \Phi_i(x)$ where $\Phi_i(x)$ mod $\mathfrak{p} = \phi_i(x)$, $\phi_i(x)$ and $\Phi_i(x)$ have the same degrees.

It can be shown that there do exist fields k on which an archimedian valuation $\mathfrak{p}$ is defined such that Hensel's criterion holds for the polynomials with coefficients in the maximal order of k although the fields under consideration are not perfect. According to A. Ostrowski we shall term such fields *relatively perfect.* The validity of Hensel's criterion implies that the valuation

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[6] For general statements about p-adic number fields see e. g. M. Deuring, *Algebren* (Berlin, 1935), Chap. VI, § 10. This tract will be quoted by D.

[7] A. Ostrowski, "Untersuchungen zur arithmetischen Theorie der Körper," *Mathematische Zeitschrift*, vol. 39 (1934).

$\mathfrak{p}$ of any relatively perfect field k possesses a unique extension $\mathfrak{P}(K)$ corresponding to each algebraic extension K of k. The value of an element $A \varepsilon K$ is given by $[K:k]^{-1}v(\mathfrak{N}A)$ where $\mathfrak{N}A$ denotes the norm of A taken from K to k. The integers of K are uniquely characterized by the fact that their norms are $\mathfrak{p}$-adic integers.

DEFINITION. *A subfield k' of a field k on which a valuation $\mathfrak{p}$ is defined is said to be everywhere dense in k if*

(I) *the valuation $\mathfrak{p}$ of k induces in k' a non-trivial valuation such that*

$$\Gamma(k) = \Gamma(k'),$$

(II) *the residue class fields $\mathbf{k}$ and $\mathbf{k}'$ coincide,*

(III) *k and k' are relatively perfect.*

THEOREM 1. *If k' is an everywhere dense subfield of k then every finite separable algebraic extension K of k is the join of a suitable extension K' of k' with k; moreover $[K:k] = [K':k']$.*

Proof. The fields k and k' obviously have the same $\mathfrak{p}$-adic closure $\bar{k}$. Let us assume that we proved the theorem for the pairs of fields $k, \bar{k}$ and $k', \bar{k}$. Then any field $\bar{K}$ over $\bar{k}$ has the form

$$\bar{K} = \bar{k} \cdot k(A) \text{ and}$$
$$\bar{K} = \bar{k} \cdot k'(A')$$

where $[\bar{K}:\bar{k}] = [A:k] = [A':k']$. Since also $[A':\bar{k}] = [A':k']$ we find

$$[A':k] = [A':k'] = [A:k].$$

Consequently

$$k(A) = k(A') = kk'(A').$$

Thus it suffices to prove the theorem under the assumption that k is a perfect $\mathfrak{p}$-adic field.

We consider the polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ of fixed degree n with coefficients as elements of a topological space $S_n(k)$ where we define

$$\delta(f(x), g(x)) = C^{-v(f(x)-g(x))}, \qquad C > 1$$

and

$$v(f(x)) = \min(v(a_0), \cdots, v(a_i), \cdots, v(a_n)).^{8}$$

The function $v(f(x))$ has the following properties

[8] F. K. Schmidt. "Mehrfach perfekte Körper," *Mathematische Annalen*, vol. **108** (1933).

$$v(f(x)+g(x)) \geqq \min(v(f(x)), v(g(x)))$$
$$v(f(x)g(x)) \geqq v(f(x)) + v(g(x)).$$

It can be proved that two polynomials $f(x)$, $g(x)$ in $S_n(k)$ have the same type of decomposition relative to k if $v(f(x)-g(x))$ is sufficiently large. We shall make use of this statement later on.

Let $g(x)$ be an arbitrary polynomial of $S_n(k)$ then

$$g(x+t) = g(x) + tD^{(1)}g(x) + t^2D^{(2)}g(x) + \cdots + t^iD^{(i)}g(x) + \cdots + t^nD^{(n)}g(x)$$

where the differential operator $D^{(i)}$ is defined by

$$D^{(i)}g(x) = \frac{d^ig(x)}{dx^i}\cdot\frac{1}{i!}.$$

It can be shown that such formal derivatives are also defined if the underlying field of coefficients k has a characteristic $\chi \neq 0$.[9]

Now let K be a finite separable algebraic extension of k. We always can assume that K is generated by a primitive element A such that its irreducible equation $f(x)$ has the form

$$a_0 + a_1x + \cdots + a_ix^i + \cdots + a_{n-1}x^{n-1} + x^n$$

where $a_0, \cdots, a_{n-1}$ are $\mathfrak{p}$-adic integers. Since $f(x)$ is a separable polynomial we have $D^{(1)}f(A) \neq 0$, moreover

$$v(D^{(1)}f(A)) = \mathrm{A} \geqq 0$$

for A is an integer of K.[10]

Now let a_i be an arbitrary coefficient of $f(x)$ then there exist elements a'_i in the everywhere dense subfield k' such that

$$v(a_i - a'_i) > \Delta > 2\mathrm{A} \geqq 0$$

where Δ is arbitrarily large but fixed. Let us then choose such a set of numbers a'_i and let us consider the polynomial

$$g(x) = \sum_{i=0}^{n} a_i'x^i$$

of degree n in k'. Then

$$v(f(x)-g(x)) > \Delta.$$

[9] H. Hasse, "Noch eine Begründung der Theorie der höheren Differentialquotienten in einem algebraischen Funktionenkörper einer Unbestimmten," *Crelle*, vol. 177 (1937).

[10] Cf. M. Moriya, "Klassenkörpertheorie im Kleinen für die unendlichen algebraischen Zahlkörper," *Journal of the Faculty of Science Hokkaido Imperial University Sapporo* (Japan), ser. I, vol. 5 (1936).

Since the coefficients a_i of $f(x)$ are integers the relation $v(f(x) - g(x)) > \Delta$ implies that the a_i' are integers of k'; $g(x)$ obviously is also a separable polynomial.

We have

$$v(g(A) - f(A)) = v(g(A)) > \Delta$$

when we substitute for the undeterminate x the root A.

Next we show that $D^{(1)}g(A) \neq 0$. Namely, $v(g(x) - f(x)) > \Delta$ implies that also $v(D^{(1)}g(x) - D^{(1)}f(x)) > \Delta$. Since A is an integer we can infer

$$v(D^{(1)}g(A) - D^{(1)}f(A)) > \Delta.$$

If we assume that $v(D^{(1)}g(A)) \neq v(D^{(1)}f(A))$ then

$$v(D^{(1)}g(A) - D^{(1)}f(A)) = \min(D^{(1)}g(A), D^{(1)}f(A)) > \Delta > \mathbf{A}$$

in contradiction to $v(D^{(1)}f(A)) = \mathbf{A} < \Delta$.

Hence

$$\infty > v(D^{(1)}f(A)) = v(D^{(1)}g(A)) = \mathbf{A}, \text{ or } D^{(1)}g(A) \neq 0.$$

Consequently we can solve the equation

$$g(A) + t_1 D^{(1)}g(A) = 0$$

with respect to t_1.

Since $v(g(A)) > \Delta > 2\mathbf{A}$ we can put

$$v(g(A)) = 2\mathbf{A} + \mathbf{E}$$

where $\mathbf{E} > 0$.

Then

$$v(t_1) = v(g(A)) - v(D^{(1)}g(A)) = \mathbf{A} + \mathbf{E} > \mathbf{A}$$

for

$$v(D^{(1)}g(A)) = \mathbf{A} < \Delta.$$

Consider next the integral element

$$A_1 = A + t_1$$

and form

$$g(A_1) = g(A + t_1) = g(A) + t_1 D^{(1)}g(A) + \cdots + t_1^n D^{(n)}g(A).$$

The coefficients of the derivatives $D^{(i)}g(x)$ are easily seen to be integral. Consequently $D^{(i)}g(A)$ are integers for A was assumed to be an integer over k. Therefore we get

$$v(t_1^i D^{(i)}g(A)) \geqq i v(t_1) \geqq i(\mathbf{A} + \mathbf{E}) \geqq 2(\mathbf{A} + \mathbf{E})$$

provided that $n \geqq i \geqq 2$. Hence

$$v(g(A_1)) = 2(\mathbf{A} + \mathbf{E}).$$

Putting $v(g(A_1)) = 2\mathbf{A} + \mathbf{E}_1$ we have

$$\mathbf{E}_1 \geqq 2\mathbf{E} \quad \text{and} \quad v(g(A_1)) = 2\mathbf{A} + \mathbf{E}_1 > 2\mathbf{A} + \mathbf{E} = v(g(A)).$$

We find

$$v(D^{(1)}f(A_1)) = v(D^{(1)}f(A + t_1)) = \mathbf{A}$$

for $v(D^{(1)}f(A)) = \mathbf{A}$ and $v(t_1) > \mathbf{A}$.

As before we can prove that

$$v(D^{(1)}g(A_1)) = \mathbf{A},$$

hence

$$D^{(1)}g(A_1) \neq 0,$$

and we can determine t_2 from

$$g(A_1) + t_2 D^{(1)}g(A_1) = 0.$$

Moreover,

$$\begin{aligned} v(t_2) &= v(g(A_1)) - v(D^{(1)}g(A_1)) \\ &= 2\mathbf{A} + \mathbf{E}_1 - \mathbf{A} = \mathbf{A} + \mathbf{E}_1 \geqq \mathbf{A} + 2\mathbf{E}. \end{aligned}$$

Introducing the integer $A_2 = A_1 + t_2$ we find

$$v(g(A_2)) \geqq 2(\mathbf{A} + \mathbf{E}_1);$$

if we put

$$v(g(A_2)) = 2\mathbf{A} + \mathbf{E}_2,$$

then

$$\mathbf{E}_2 \geqq 2\mathbf{E}_1 \geqq 2^2\mathbf{E}.$$

Again

$$v(D^{(1)}g(A_2)) = \mathbf{A}.$$

Repeating these arguments we can construct an element t_i such that

$$v(t_i) \geqq \mathbf{A} + 2^{i-1}\mathbf{E} \quad \text{and} \quad v(g(A_i)) = 2\mathbf{A} + \mathbf{E}_i \geqq 2\mathbf{A} + 2^i\mathbf{E}$$

where

$$A_i = A_{i-1} + t_i.$$

Let us consider the sequence of elements

$$A, A_1 = A + t_1, \cdots, A_i = A + t_1 + \cdots + t_i, \cdots.$$

These elements form a fundamental sequence of algebraic quantities over k for

$$v(A_i - A_{i+j}) \geqq \mathbf{A} + 2^i\mathbf{E}$$

if $j \geqq 1$.

Since K is a perfect field the sequence $\{A_i\}$ has a limit $\bar{A}$ in K. According to our construction we have

$$\lim_{i\to\infty} v(g(A_i)) = \infty,$$

hence the sequence $\{g(A_i)\}$ converges to 0 in K, or ultimately

$$g(\bar{A}) = 0 \text{ in } K.$$

Since the polynomial $g(x)$ is chosen in k' the root $\bar{A}$ of $g(x) = 0$ is algebraic over k'. Moreover, $g(x)$ is irreducible in k according to the remark at the beginning of the proof; a fortiori $g(x)$ is irreducible in k'. Consequently $[\bar{A}:k'] = [\bar{A}:k] = [K:k]$ or

$$K = k(\bar{A}) = kk'(\bar{A}).$$

In particular, $k'(\bar{A})$ is a normal extension of k' if K is normal over k'.

The theorem we just proved has an analogue for the normal simple algebras over k and k'.

THEOREM 2. *If k' is an everywhere dense subfield of k then each normal algebra $\mathfrak{A}$ over k is similar to the direct product of a normal simple algebra $\mathfrak{A}'$ over k' with the field k.*

Proof. Let $\mathfrak{D}$ be an arbitrary normal division algebra over k. Then $\mathfrak{D}$ can be represented uniquely as the direct product $\mathfrak{D}' \times \mathfrak{D}^{(\chi)}$ of a division algebra $\mathfrak{D}'$ whose degree is relatively prime to the characteristic χ and another division algebra $\mathfrak{D}^{(\chi)}$ whose degree is a power of χ. Consequently, it suffices to prove the theorem for the components separately. As in Theorem 1 it suffices to prove the theorem in the case that k is perfect.

Case i. $\mathfrak{D} = \mathfrak{D}'$.

The division algebra $\mathfrak{D}$ can be represented by a crossed product $(a_{\sigma,\tau}, K/k)$ where K denotes a suitable normal extension of k whose Galois group consists of the elements $\{\sigma, \tau, \cdots\}$. Since the degree of $\mathfrak{D}$ is relatively prime to χ we have

$$a^e{}_{\sigma,\tau} = c_\sigma c_\tau{}^\sigma c_{\sigma\tau}^{-1}$$

where $(e, \chi) = 1$. Consequently there exists the least normal extension N over k which contains K and the quantities $c_\sigma{}^{1/e}, c_\tau{}^{1/e}, \cdots$. Let $\{\Sigma, \mathrm{T}, \cdots\}$ be the Galois group of N/k then

$$\mathfrak{D} \sim (a_{\sigma,\tau}, K/k) \sim (\zeta^{(e)}_{\Sigma,\mathrm{T}}, N/k)$$

where $\zeta^{(e)}_{\Sigma,\mathrm{T}}$ are suitable e-th roots of unity.[11] Now we can apply Theorem 1,

[11] D. Chap. V, § 7.

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the field N has the form $N = N' \cdot k$ where $[N' : k'] = [N : k]$ and where the respective Galois groups are isomorphic. Observing that the quantities $\zeta^{(e)}_{\Sigma,\mathrm{T}}$ are absolutely algebraic, we can construct the algebra

$$(\zeta^{(e)}_{\Sigma,\mathrm{T}}, N'/k') \text{ over } k',$$

obviously

$$\mathfrak{D} \sim (\zeta^{(e)}_{\Sigma,\mathrm{T}}, N'/k') \times k.$$

Case ii. $\mathfrak{D} = \mathfrak{D}^{(\chi)}$.

According to a theorem of A. A. Albert each division algebra of the type we have to consider, is similar to the direct product of cyclic crossed products. Consequently, it suffices to prove the theorem for cyclic algebras $(a, Z/k)$. Let Z' be the separable extension of k' such that $Z = Z'k$. We have to construct a factor set a' in k' such that

$$(a', Z'/k') \times k \sim (a, Z/k).$$

Since k was assumed to be perfect, the element a of k is the limit of a fundamental sequence $\{a'_i\}$ of elements a'_i in k'. Using the fundamental properties on the convergence of $\mathfrak{p}$-adic sequences and series, we find that

$$a = \sum_{i=i_0}^{\infty} (a'_i - a'_{i-1})$$

where i_0 denotes the least subscript such that

$$v(a'_{i_0}) = v(a).$$

The section $a' = \sum_{i=i_0}^{m} (a'_i - a'_{i-1})$ denotes an element of k' whose value $v(a')$ coincides with $v(a)$. The element $aa'^{-1} = b$ is a unit of k and

$$b = 1 + c$$

where $v(c) > 0$. The usual arguments concerning $\mathfrak{p}$-adic valuations yield that for any prescribed $N > 0$ in the value group $\Gamma(k)$ there exists a number $m = m(N)$ *such that*

$$v(c) \geqq N.$$

Now let us consider the expression

$$1 + y = \mathfrak{N}(1 + xA) = 1 + t(A)x + \cdots$$

where A denotes a suitable primitive element of Z/k whose trace $t(A)$ does not vanish. The element x can be expressed in terms of a convergent power series in y provided that $v(y)$ is sufficiently large, say $v(y) \geqq N$. Consequently the element $1 + y$ is a norm. We apply this result to the unit $b = 1 + c$

where we assume that $v(c) \geqq N$, i.e. that m is sufficiently large. Then $b = \mathfrak{N}B$ with $B \varepsilon Z$, hence

$$(a, Z/k) = (a'b, Z/k) \sim (a', Z/k) \times (b, Z/k)$$
$$\sim (a', Z/k).$$

Again the field Z has the form $Z'k$ where Z' is cyclic over k', hence obviously

$$(a', Z'/k') \times k \sim (a, Z/k).$$

Remark. In Theorems 1 and 2 we can omit the assumption that the subfield k' be relatively perfect.

2. Specialization of the value group $\Gamma(k)$. Let K be an arbitrary normal extension of the relatively perfect field k and let $G = \{\sigma, \tau, \cdots\}$ be its Galois group. Since k is assumed to be relatively perfect the uniquely determined prime ideal $\mathfrak{P}$ of K is left invariant by all substitutions of the Galois group. G contains a set of subgroups which are determined by the arithmetical properties of the extension K.[12]

I) all substitutions $\sigma \varepsilon G$ such that

$$\epsilon^\sigma = \epsilon\eta \text{ with } \eta \equiv 1 \pmod{\mathfrak{P}}$$

for all units ϵ of K, form an invariant subgroup I of G, the so-called *inertial group.*

II) All substitutions $\tau \varepsilon I$ such that

$$a^\tau = a\eta \text{ with } \eta \equiv 1 \pmod{\mathfrak{P}}$$

holds for all elements a of K form an invariant subgroup R of G the so-called *ramification group.*

The fields $K(I)$ and $K(R)$, which belong to the groups I and R respectively, are called the inertial and ramification fields respectively. If we assume that the degree $[K:k]$ is relatively prime to the characteristic χ then it can be proved that $K(I)$ is the maximal subfield U of K such that

$$\Gamma(U) = \Gamma(k) \text{ and } \boldsymbol{U} = \boldsymbol{K}, \text{ i.e.}$$

$K(I)$ is the maximal unramified extension lying in K, when an unramified extension U of k is characterized by the relations

$$\Gamma(U) = \Gamma(k) \text{ and } [U:k] = [\boldsymbol{U}:\boldsymbol{k}].$$

If U is in particular a normal extension of k then its Galois group induces the Galois group of $\boldsymbol{U}$ over $\boldsymbol{k}$.

[12] W. Krull, "Allgemeine Bewertungstheorie," *Crelle*, vol. 167 (1931).

The inertial field $K(I)$ is in any case uniquely determined as the maximal unramified subfield if the field of residue classes $\boldsymbol{k}$ is algebraically perfect.

The ramification field $K(R)$ is always an *abelian* extension of K whose Galois group I/R is isomorphic with the factor group $\Gamma(K(R))/\Gamma(K(I))$. If K is a proper algebraic extension of $K(R)$ then $[K:K(R)]$ is a power of the characteristic. Consequently $K(R)=K$ if $([K:k],\chi)=1$, and hence K is an abelian extension of $K(I)$ where

$$I \cong \Gamma(K)/\Gamma(K(I)) = \Gamma(K)/\Gamma(k).$$

LEMMA 1. *If the residue class field* $\boldsymbol{k}$ *of the relatively perfect field* k *is algebraically closed then each extension* K *of* k *whose degree is relatively prime to the characteristic* χ *is abelian.*

Proof. Let K be an arbitrary algebraic extension of k whose degree is relatively prime to χ. The field K is contained in a least normal extension N of k. Since $\boldsymbol{k}$ is assumed to be algebraically closed we have $N(I)=k$. The results concerning the ramification field $N(R)$ imply that

$$K \subseteq N(R)$$

for $([K:k],\chi)=1$ and $[N:N(R)]$ is at most a power of χ. Consequently, K is an abelian extension of k for the Galois group of $N(R)$ over k is isomorphic with $\Gamma(N(R))/\Gamma(k) \cong \Gamma(K)/\Gamma(k)$.

We now wish to specialize the value group $\Gamma(k)$ in order to be able to obtain more precise results concerning the theory of algebras which we have to develop later on. We assume that $\Gamma(k)$ contains the additive group of all rational numbers P and a finite number of infinite cyclic groups $\theta_i\Gamma^{(0)}$ where θ_i $(i=1,\cdot\cdot\cdot,\lambda)$ denote a set of rationally independent real numbers and where $\Gamma^{(0)}$ denotes the additive group of all integers:

$$\Gamma(k)=\Gamma=\{\mathrm{P},\theta_1\Gamma^{(0)},\cdot\cdot\cdot,\theta_\lambda\Gamma^{(0)}\}.$$

Each element $\alpha\,\varepsilon\,\Gamma$ can then uniquely be represented in the form

$$\alpha=\beta+\sum_{i=1}^{\lambda}\gamma_i\theta_i$$

where $\beta\,\varepsilon\,\mathrm{P}$ and $\gamma_i\,\varepsilon\,\Gamma^{(0)}$.

LEMMA 2. *Let* k^* *and* $\boldsymbol{k}^*$ *denote the multiplicative groups of* k *and* $\boldsymbol{k}$ *respectively. If the integer* n *is relatively prime to the characteristic* χ *of* k *then*

$$[k^*:k^{*n}]=[\Gamma/n\Gamma:1][\boldsymbol{k}^*:\boldsymbol{k}^{*n}]$$

if $[\boldsymbol{k}^*:\boldsymbol{k}^{*n}]$ *is finite.*

Proof. Let us fix λ elements $t_1,\cdots,t_\lambda$ in k such that

$$v(t_i)=\theta_i \qquad (i=1,\cdots,\lambda).$$

Then each element a of k^* can be written in the form

$$a=\prod_{i=1}^{\lambda} t_i^{\gamma_i} b$$

where

$$v(a)=\beta+\sum_{i=1}^{\lambda}\gamma_i\theta_i \quad \text{and} \quad v(b)=v(a\prod_{i=1}^{\lambda}t_i^{-\gamma_i})=\beta\,\varepsilon\,\mathrm{P}.$$

All elements $b\,\varepsilon\,k^*$ whose values $v(b)$ lie in the rational component P form a multiplicative subgroup B of k^*, and obviously $k^*=\langle\{t_1\},\cdots,\{t_\lambda\},B\rangle$ where $\{t_i\}$ denote the infinite cycles generated by the elements t_i; $\langle\ldots\rangle$ denotes the join of $\{t_1\},\cdots,\{t_\lambda\}$ and B. Since 1 is the only element of k^* contained in both $\langle\{t_1\},\cdots,\{t_\lambda\}\rangle$ and B we see that

$$k^*=\langle\{t_1\},\cdots,\{t_\lambda\}\rangle\times B.$$

We find $k^{*n}=\langle\{t_1\}^n,\cdots,\{t_\lambda\}^n\rangle\times B^n$ as a consequence of

$$a^n=\prod_{i=1}^{\lambda}t_i^{\gamma_i n}b^n.$$

Consequently,

$$\begin{aligned}k^*/\langle\{t_1\},\cdots,\{t_\lambda\}\rangle^n\times B &\cong \{t_1\}/\{t_1\}^n\times\cdots\times\{t_\lambda\}/\{t_\lambda\}^n\times B/B\\ &\cong \{t_1\}/\{t_1\}^n\times\cdots\times\{t_\lambda\}/\{t_\lambda\}^n.\end{aligned}$$

We have

$$\begin{aligned}[k^*:k^{*n}] &= [k^*/\langle\{t_1\},\cdots,\{t_\lambda\}\rangle^n\times B:1][\langle\{t_1\},\cdots,\{t_\lambda\}\rangle^n\times B/k^{*n}:1]\\ &= n^\lambda[\langle\{t_1\},\cdots,\{t_\lambda\}\rangle^n\times B/k^{*n}:1]\\ &= n^\lambda[\langle\{t_1\},\cdots,\{t_\lambda\}\rangle^n\times B/\langle\{t_1\},\cdots,\{t_\lambda\}\rangle^n\times B^n:1]\\ &= n^\lambda[\langle\{t_1\},\cdots,\{t_\lambda\}\rangle^n\times B/\langle\{t_1\},\cdots,\{t_\lambda\}\rangle^n/\langle\{t_1\},\cdots,\{t_\lambda\}\rangle^n\\ &\qquad\times B^n/\langle\{t_1\},\cdots,\{t_\lambda\}\rangle^n:1]\\ &= n^\lambda[B/B^n:1].\end{aligned}$$

We next have to reduce the index $[B/B^n:1]$ under the assumption that $(n,\chi)=1$. Let b be an arbitrary element of B, then $v(b)=\beta\,\epsilon\,\mathrm{P}$. Consequently, there exists at least one element c in k such that $v(c)=\beta/n$. Therefore $b=c^n\epsilon$ where ϵ is a suitable unit. Hence $B=\langle B^n,\epsilon\rangle$ where ϵ stands for the group of units in k. Therefore

$$\begin{aligned}B/B^n &= \langle B^n,\epsilon\rangle/\langle B^n,\epsilon^n\rangle=\langle B^n,\epsilon\rangle/B^n\\ &\cong \epsilon/B^n\cap\epsilon=\epsilon/\epsilon^n\end{aligned}$$

since $v(\epsilon)=0$.

Now let η be a unit of k and let $\boldsymbol{a}$ be its uniquely determined residue class mod $\mathfrak{p}$. If we assume that the equation $x^n - \boldsymbol{a} = 0$ is divisible in $\boldsymbol{k}$ by a linear factor $x - \boldsymbol{b}$, i. e. $x^n - \boldsymbol{a} = (x - \boldsymbol{b})\boldsymbol{g}(x)$ such that $((x - \boldsymbol{b}), \boldsymbol{g}(x)) = 1$ then the polynomial $x^n - \eta = 0$ is divisible by a linear factor $x - \eta'$ where $\eta' \bmod \mathfrak{p} = \boldsymbol{b}$ for Hensel's criterium of reducibility holds. Consequently, each unit η which is $\equiv 1 \pmod{\mathfrak{p}}$ can be written as a n-th power.

Now let us consider $\mathfrak{p}$ as a multiplicative mapping of the group of all units ϵ upon the multiplicative group $\boldsymbol{k}^*$. Then

$$[\epsilon/\epsilon^n : 1] = [\boldsymbol{k}^* : \boldsymbol{k}^{*n}]$$

as the application of Herbrand's Lemma yields. Hence finally

$$[k^* : k^{*n}] = n^\lambda [\boldsymbol{k}^* : \boldsymbol{k}^{*n}].$$

Remark. If we assume that the residue class field $\boldsymbol{k}$ possesses for each integer n which is relatively prime to χ, exactly one cyclic extension $\boldsymbol{U}_n$ of degree n then obviously

$$[\boldsymbol{k}^* : \boldsymbol{k}^{*n}] \leqq n.$$

Namely, if $\boldsymbol{k}$ contains the n-th roots of unity then $\boldsymbol{U}_n = \boldsymbol{k}(\boldsymbol{a}^{1/n})$ where $\boldsymbol{k}^* = \boldsymbol{k}^{*n} + \boldsymbol{a}\boldsymbol{k}^{*n} + \cdots + \boldsymbol{a}^{n-1}\boldsymbol{k}^{*n}$, and if $\boldsymbol{k}$ does not contain the n-th roots of unity then surely $[\boldsymbol{k}^* : \boldsymbol{k}^{*n}] \leqq n$ for otherwise there would exist more than one cyclic extension $\boldsymbol{U}_n$ as a simple application of the Galois theory shows. Therefore we have

$$[k : k^{*n}] \leqq n^{\lambda+1}.$$

The structure of k^*/k^{*n} is given by

$$\{t_1\} \times \cdots \times \{t_\lambda\}/\{t_1\}^n \times \cdots \times \{t_\lambda\}^n \times \boldsymbol{k}^*/\boldsymbol{k}^{*n} \cong \Gamma(k)/n\Gamma(k) \times \boldsymbol{k}^*/\boldsymbol{k}^{*n}.$$

COROLLARY. *Let K be an arbitrary algebraic extension of k whose degree is relatively prime to χ then obviously $k^{*n} \subseteq \mathfrak{N}K^*$ where $\mathfrak{N}K^*$ denotes the multiplicative group of the norms of all elements in K^*. Then obviously*

$$[k^*/\mathfrak{N}K^* : 1] \leqq n^{\lambda+1}$$

as the application of the preceding remark shows.

LEMMA 3. *If the field of residue classes $\boldsymbol{k}$ is algebraically closed then the largest abelian extension K of k which has exponent n — $(n, \chi) = 1$—is given by $k(t_1^{1/n}, \cdots, t_\lambda^{1/n})$.*

Proof. According to the theory of radical fields we find that the largest abelian extension K of exponent n is given by adjoining the n-th roots of a

set of representatives of k^*/k^{*n}. Hence $[K:k] = n^\lambda$ according to Lemma 2. Now $t_1, \cdots, t_\lambda$ form a complete set of representatives of k^*/k^{*n}, consequently $K = k(t_1^{1/n}, \cdots, t_\lambda^{1/n})$. Incidentally, we find that each element b of k^* whose value is rational is a n-th power.

We next wish to investigate the structural properties of the class group of all normal simple algebras $\mathfrak{G}(k)$ over k. Now each division algebra $\mathfrak{D}$ with k as center is the direct product of a division algebra $\mathfrak{D}'$ whose degree is relatively prime to χ and of a division algebra $\mathfrak{D}^{(\chi)}$ whose degree is a power of χ. Consequently, the group $\mathfrak{G}(k)$ is the direct product of two subgroups $\mathfrak{G}'(k)$ and $\mathfrak{G}^{(\chi)}(k)$ where all classes in $\mathfrak{G}'(k)$ can be represented by division algebras of the type $\mathfrak{D}'$ and all algebras in $\mathfrak{G}^{(\chi)}(k)$ are representable by division algebras of type $\mathfrak{D}^{(\chi)}$. Therefore it suffices to investigate the structural properties of the two components.

3. The structure of $\mathfrak{G}^{(\chi)}(k)$. In order to describe the structure of $\mathfrak{G}^{(\chi)}(k)$ we can apply the results of A. A. Albert and other authors concerning χ-algebras.[13] With the help of this additive theory one proves

THEOREM 3. *Each division algebra in $\mathfrak{G}^{(\chi)}(k)$ is similar to the direct product of cyclic algebras.*

In general one cannot say more about the structure of $\mathfrak{G}^{(\chi)}(k)$ since it depends on the structure of k as an additive group, i. e. it is closely related to the invariantive characterization of general perfect fields. However, if we assume that the field of residue classes $\boldsymbol{k}$ admits for each degree χ^ν $(\nu = 1, 2, \cdots)$ exactly one cyclic extension $\boldsymbol{U}_\nu$ we can say more about $\mathfrak{G}^{(\chi)}(k)$.

THEOREM 4. *$\mathfrak{G}^{(\chi)}(k)$ contains a subgroup $\mathfrak{G}_u{}^{(\chi)}(k)$ which consists of the totality of all classes which are split by unramified fields U_ν $(\nu = 1, 2, \cdots)$. $\mathfrak{G}_u{}^{(\chi)}(k)$ is isomorphic with the direct sum of λ additive groups of type* $\Gamma^{(0)}\chi^{-\infty}$ mod 1; *where* $\Gamma^{(0)}\chi^{-\infty}$ mod 1 *denotes the additive group of all fractions* mod 1 *whose denominators are powers of χ.*

Proof. Let $U^{(\chi)}$ be the infinite unramified extension of k containing all the unramified cyclic fields U_ν which are uniquely determined by the extensions $\boldsymbol{U}_\nu$ over $\boldsymbol{k}$. If $\{\mathfrak{D}_1\}$ and $\{\mathfrak{D}_2\}$ are any two classes of $\mathfrak{G}^{(\chi)}(k)$ which are split by an unramified field then obviously

[13] A. A. Albert, "Normal division algebras of degree p^e over fields of characteristic p," *Transactions of the American Mathematical Society*, vol. 39 (1936); T. Nakayama, "Über Algebren über einem Körper von der Primzahlcharakteristik, II," *Proceedings of the Imperial Academy of Tokyo*, vol. 12 (1937).

$$\{\mathfrak{D}_1\} \times U^{(\chi)} \sim \{\mathfrak{D}_2\} \times U^{(\chi)} \sim U^{(\chi)}$$

whence

$$\{\mathfrak{D}_1\} \times \{\mathfrak{D}_2\} \times U^{(\chi)} \sim U^{(\chi)}$$

and

$$\{\mathfrak{D}_i^{-1}\} \times U^{(\chi)} \sim U^{(\chi)}$$

i. e. all algebras which have unramified splitting fields form a group $\mathfrak{G}_u^{(\chi)}(k)$.

In order to determine the structure of $\mathfrak{G}_u^{(\chi)}(k)$ it suffices to describe the structure of the subgroup consisting of all algebras which are split by a fixed unramified field U_ν for the fields U_ν form a linearly ordered tower of successive algebraic extensions with the limit $U^{(\chi)}$. The structure of these groups $\mathfrak{G}_{U_\nu}^{(\chi)}(k)$ is determined as follows. Any algebra $\mathfrak{A}$ in $\mathfrak{G}_{U_\nu}^{(\chi)}(k)$ can be represented as a cyclic crossed product $(a, U_\nu/k)$. Let us fix elements t_i $(i=1,\cdots,\lambda)$ in k such that $v(t_i)=\theta_i$, then

$$v(a)=\alpha=\beta+\sum_{i=1}^{\lambda}\gamma_i\theta_i \text{ with } \beta \in \mathrm{P}$$

implies

$$a=b^{\chi^\nu}\prod_{i=1}^{\lambda}t_i^{\gamma_i}\epsilon$$

where b denotes an arbitrary element in k whose value $v(b)$ is equal to $\beta\chi^{-\nu}$. Hence

$$(a, U_\nu/k) \sim (b^{\chi^\nu}, U_\nu/k) \times (t_1, U_\nu/k)^{\gamma_1} \times \cdots \times (t_\lambda, U_\nu/k)^{\gamma_\lambda} \times (\epsilon, U_\nu/k)$$
$$\sim (t_1, U_\nu/k)^{\gamma_1} \times \cdots \times (t_\lambda, U_\nu/k)^{\gamma_\lambda} \times (\epsilon, U_\nu/k).$$

We next prove that $(\epsilon, U_\nu/k) \sim k$.

First we remark that there do not exist cyclic algebras $(\epsilon, U_\nu/k)$ whose exponent is equal to the degree. Namely, suppose that the exponent and degree of $(\epsilon, U_\nu/k)$ coincide. Then $(\epsilon, U_\nu/k)=\mathfrak{D}$ is a division algebra. Consequently, the valuation $\mathfrak{p}$ of the ground field k can be extended to the algebra $\mathfrak{D}$ as we shall prove later on page 691. Hence the algebra of residue classes belonging to $\mathfrak{D}$ is a division algebra $\boldsymbol{D}$ over $\boldsymbol{k}$. Since the maximal order of $\mathfrak{D}$ contains the order which is given by all sums $\Sigma u^i\omega_i$ where ω_i are integers in U_ν, it follows that $\boldsymbol{D}$ has the center $\boldsymbol{k}$ and is explicitly given as $(\epsilon \bmod \mathfrak{p}, \boldsymbol{U}_\nu/\boldsymbol{k})$. Such a division algebra $\boldsymbol{D}$ cannot exist since we suppose that the field of residue classes $\boldsymbol{k}$ is algebraically perfect. Consequently $\mathfrak{D} \sim k$. Assume now that $(\epsilon, U_\nu/k)$ is an arbitrary algebra whose factor set is a unit ϵ of k. Suppose that the exponent s_1 of $(\epsilon, U_\nu/k)$ is greater than 1. We already proved that s_1 has to be a proper divisor of $[U_\nu : k]$. We have

$$(\epsilon^{s_1}, U_\nu/k) \sim (\epsilon, U_\nu^{(1)}/k) \sim k$$

where $U_\nu^{(1)}$ denotes the uniquely determined cyclic subfield of U_ν such that $[U_\nu : U_\nu^{(1)}] = s_1$. Consequently, $\epsilon = \mathfrak{N}_1\epsilon_1$ where ϵ_1 is a suitable unit of $U_\nu^{(1)}$ and where $\mathfrak{N}_1$ denotes the norm taken from $U_\nu^{(1)}$ to k. Next consider the algebra $(\epsilon_1, U_\nu/U_\nu^{(1)})$. Its exponent s_2 has to be a proper divisor of the degree $[U_\nu : U_\nu^{(1)}]$ as a repetition of the argument at the beginning of the proof shows, $U_\nu^{(1)}$ is also algebraically perfect. If s_2 should be greater than 1, then

$$(\epsilon_1^{s_2}, U_\nu/U_\nu^{(1)}) \sim (\epsilon_1, U_\nu^{(2)}/U_\nu^{(1)}) \sim U_\nu^{(1)}$$

where $U_\nu^{(2)}$ is the uniquely determined subfield of U_ν such that $[U_\nu : U_\nu^{(2)}] = s_2$. Hence $\epsilon_1 = \mathfrak{N}_2\epsilon_2$ where ϵ_2 is a unit of $U_\nu^{(2)}$ and where $\mathfrak{N}_2$ denotes norm taken from $U_\nu^{(2)}$ to $U_\nu^{(1)}$. We can repeat this argument only a finite number of times since the degree $[U_\nu : k]$ is finite. Observing that the norm is a transitive multiplicative function we find that $\epsilon = \mathfrak{N}E$ with a suitable unit E of U_ν. This statement contradicts the assumption that $(\epsilon, U_\nu/k) \not\sim k$, therefore we must have $(\epsilon, U_\nu/k) \sim k$ in general.

Consequently, each algebra in $\mathfrak{G}_{U_\nu}{}^{(\chi)}(k)$ is similar to the direct product of λ algebras $(t_i, U_\nu/k)^{\gamma_i}$. The elements t_i are not norms of elements in U_ν for U_ν is an unramified extension of k. Hence

$$k^*/\mathfrak{N}U^*_\nu \cong \Gamma/\chi^\nu\Gamma \cong Z_1(\chi^\nu) \times \cdots \times Z_\lambda(\chi^\nu),$$

where $Z_i(\chi^\nu)$ denote λ cyclic groups of order χ^ν. Since each of the groups $Z_i(\chi^\nu)$ is isomorphic with the additive group of all fractions mod 1 whose denominators are divisors of χ^ν, we find for $\nu \to \infty$ that $\mathfrak{G}_{U_\nu}{}^{(\chi)}(k)$ is isomorphic with the direct sum of λ additive groups $\Gamma^{(0)}\chi^{-\infty}$ mod 1 consisting of all rational numbers mod 1 whose denominators are powers of χ.

The group $\mathfrak{G}_u{}^{(\chi)}(k)$ is in general a proper subgroup of $\mathfrak{G}^{(\chi)}(k)$ and it is not a direct factor of $\mathfrak{G}^{(\chi)}(k)$ as can easily be seen by examples. We wish to show by two examples that $\mathfrak{G}_u{}^{(\chi)}(k) \subset \mathfrak{G}^{(\chi)}(k)$; these examples will also illustrate how in general the additive structure and algebraic structure of the given ground field k influence the structure of $\mathfrak{G}^{(\chi)}(k)$. It is obvious that $\mathfrak{G}_u{}^{(\chi)}(k) \subset \mathfrak{G}^{(\chi)}(k)$ if $\lambda \geqq 2$. Namely, the algebra (t_2, t_1) with the generating relations

$$u^\chi = t_2,\ y^\chi - y = t_1,\ u^{-1}yu = y + 1$$

is not split if we extend k to the maximal unramified field $U^{(\chi)}$. In order to prove $\mathfrak{G}^{(\chi)}(k) \supset \mathfrak{G}_u{}^{(\chi)}(k)$ for a field k it is necessary and sufficient to show that $\mathfrak{G}^{(\chi)}(U) \neq 1$ where $U^{(\chi)}$ denotes the maximal unramified field over k. Let us assume for the present that $k = U^{(\chi)}$. If one considers infinite algebraic extensions of fields of formal power series of one variable with coefficients in an

algebraically closed field of characteristic $\chi \neq 0$ then one finds that $\mathfrak{G}^{(\chi)}(k) = 1$. Thus one could suspect that $\mathfrak{G}^{(\chi)}(k) = 1$ if $\Gamma(k) = \mathrm{P}$, or $= \{\mathrm{P}, \theta\Gamma^{(0)}\}$. We shall give two examples of fields k whose value groups $\Gamma(k)$ have either one of the types mentioned and whose fields of residue classes are algebraically closed but for which $\mathfrak{G}^{(\chi)}(k) \neq 1$.

Let Ω be an arbitrary algebraically closed field of characteristic $\chi \neq 0$. Consider then the field $\Omega\{t\}$ which consists of all formal power series in t with coefficients in Ω. According to well-known results the field $\Omega\{t\}$ is perfect with respect to a discrete archimedian valuation $\mathfrak{p}$ and Ω is the associated field of residue classes. We adjoin to $\Omega\{t\}$ all elements $t^{1/n}$ where $(n, \chi) = 1$, the resulting field $\Omega\{t\}(\langle t^{1/n}\rangle)$ is an infinite algebraic extension of $\Omega\{t\}$ whose value group $\Gamma(\Omega\{t\}(\langle t^{1/n}\rangle))$ consists of all rational numbers whose denominators are relatively prime to χ. Again Ω is the field of residue classes. Next we adjoin to $\Omega\{t\}$ all solutions of the cyclic equations of degree χ^ν which are given by the roots of

$$\dot{y}_\nu{}^\chi \dot{-} \dot{y}_\nu = \dot{a}_\nu$$

where $\dot{y}_\nu = (y_1, \cdots, y_\nu)$ and $\dot{a}_\nu = (t^{-1}, \cdots, t^{-1})$.[14] The resulting cyclic field Z_ν has degree χ^ν over $\Omega\{t\}$ and $\mathfrak{p}$ is completely ramified in Z_ν, i.e. $\mathfrak{p} = \mathfrak{p}_\nu{}^{\chi^\nu}$ where $\mathfrak{p}_\nu$ denotes the prime ideal of Z_ν. The fields Z_ν form an infinite tower of relatively cyclic fields

$$\Omega\{t\} < Z_1 < \cdots < Z_i < Z_{i+1} < \cdots,$$

the join of these fields is an infinite cyclic extension Z_∞ of $\Omega\{t\}$ whose degree is equal to χ^∞. The value group $\Gamma(Z_\infty)$ consists of all rational numbers whose denominators are powers of χ, again Ω is the field of residue classes. The join of $\Omega\{t\}(\langle t^{1/n}\rangle)$ and Z_∞ is an infinite separable abelian extension A_∞ of $\Omega\{t\}$ whose value group $\Gamma(A_\infty)$ consists of all rational numbers P and whose field of residue classes $\mathfrak{o}_\infty/\mathfrak{p}_\infty$ is equal to the algebraically closed field Ω.

The equation $f(x) \equiv x^\chi - x = t^{-(1+\chi)} = 0$ is not solvable in any one of the fields $\Omega\{t\}$, $\Omega\{t\}(\langle t^{1/n}\rangle)$ and Z_∞, consequently it is not solvable in A_∞. Hence a solution of $f(x) = 0$ generates a cyclic extension Z of degree χ over A_∞, Z is ramified over A_∞. We only have to prove that $f(x)$ is not solvable in Z_∞. Consider for this purpose the discriminant of the field $\Omega\{t\}(x)$, it is equal $\mathfrak{p}^{(\chi-1)(2+\chi)}$ whereas the field $\Omega\{t\}(y)$ which represents the only existing subfield Z_1 of degree χ in Z_∞ has the discriminant $\mathfrak{p}^{(\chi-1)2}$. Consequently $\Omega\{t\}(x) \neq Z_1$, hence $Z = A_\infty(x)$ has degree χ over A_∞. The field A_∞ is not perfect, let L be its perfect closure. Then

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[14] E. Witt, "Zyklische Körper und Algebren der Charakteristik p von Grade p^n," *Crelle*, vol. 176 (1937); H. L. Schmid, "Zur Arithmetik der zyklischen $\mathfrak{p}$-Körper," *Crelle*, vol. 176 (1937).

$$[ZL:L]=[Z:A_\infty]=\chi$$

according to Theorem 1.

Next we adjoin to L a transcendental element T such that either

α) $v(T)=$ irrational θ (for example an invariable element), or

β) $v(T)=$ rational, i. e. T is a pseudo limit of a pseudo convergent sequence of 2nd kind $\{a_i\}$ with elements in the algebraic closure of L.

In either case we know according to A. Ostrowski that the valuation $\mathfrak{p}_\infty$ of L is extended in a unique fashion to a valuation $\mathfrak{P}$ of $L(T)$ such that the field of residue classes of $L(T)$ with respect to $\mathfrak{P}$ is equal to Ω. For the value groups we have:

α) $\Gamma(L(T))=\{\Gamma(L),\Gamma^{(0)}\theta\}=\{\mathrm{P},\theta\Gamma^{(0)}\}$ and

β) $\Gamma(L(T))=\Gamma(L)=\mathrm{P}$, respectively.

Now let k be the $\mathfrak{P}$-adic closure of $L(T)$ then in either case $\mathfrak{o}/\mathfrak{P}=\Omega$ — $\mathfrak{o}$ the maximal order of all $\mathfrak{P}$-adic integers in k—and

α) $\Gamma(k)=\Gamma(L(T))=\{\mathrm{P},\theta\Gamma^{(0)}\}$ or

β) $\Gamma(k)=\Gamma(L(T))=\mathrm{P}$.

Since $L(T)$ is everywhere dense in k we can establish a one to one correspondence between the classes of normal simple algebras over $L(T)$ and k. We wish to prove that $\mathfrak{G}^{(\chi)}(k)\neq 1$, hence it suffices to show the existence of proper χ-algebras over $L(T)$. We shall deal with the 2 cases separately.

Case α. The join $ZL(T)$ is a cyclic ramified extension of $L(T)$ and obviously $\Gamma(ZL(T))=\Gamma(L(T))$ for $v(x)\,\varepsilon\,\mathrm{P}$. Hence

$$\chi\Gamma(ZL(T))=\mathfrak{N}\Gamma(ZL(T))=\chi\Gamma(L(T))=\{\mathrm{P},\chi\theta\Gamma^{(0)}\}$$

where $\mathfrak{N}\Gamma(ZL(T))$ denotes the value group of the norms of all elements in $ZL(T)$. Consider now the algebra $(T,ZL(T)/L(T))$ with the generating relations

$$x^\chi-x=t^{-(1+\chi)},\ u^\chi=T,\ u^{-1}xu=x+1.$$

This algebra is not similar to $L(T)$ for then T has to be norm $\mathfrak{N}T'$ of a suitable element $T'\,\varepsilon\,ZL(T)$. This would imply for the values

$$\chi v(T)=\theta\,\varepsilon\,\mathfrak{N}\Gamma(ZL(T))=\{\mathrm{P},\theta\Gamma^{(0)}\},$$

such an inclusion cannot be true according to the ramification theory. Hence $(T,ZL(T)/L(T))\not\sim L(T)$, or

$$(T,Zk/k)\not\sim k,\ \text{i. e. } G^{(\chi)}(k)\neq 1.$$

Case β. As before we consider the cyclic extension $ZL(T)$ of $L(T)$. But now we get

$${}_{\chi}\Gamma(ZL(T)) = \mathfrak{N}\Gamma(ZL(T)) = \Gamma(L(T)) = \Gamma(ZL(T)) = P.$$

The field $ZL(T)$ is ramified over $L(T)$ but we can no more measure the ramification by comparing the respective value groups.

Since it is sufficient to exhibit an example of a perfect field whose field of residue classes is algebraically closed and for which nevertheless $\mathfrak{G}^{(\chi)}(k) \neq 1$ we specialize $\chi = 2$ for sake of simplicity. Then $f(x) \equiv x^2 + x + t^{-3} = 0$. We consider the algebra $(T, ZL(T)/L(T))$ of degree 2 which is given by the generating relations

$$x^2 + x + t^{-3} = 0, \quad u^2 = T, \quad u^{-1}xu = x + 1.$$

An arbitrary element $a + bx$ of $ZL(T)$ has the characteristic equation

$$g(z) \equiv z^2 + az + a^2 + ab + b^2t^{-3} = 0,$$

hence $\mathfrak{N}(a + bx) = a^2 + ab + b^2t^{-3}$. The elements of $L(T)$ have the form $(\Sigma a_iT^i)(\Sigma c_iT^i)^{-1}$ where Σa_iT^i, Σc_iT^i are polynomials of T with coefficients in L. Suppose now that the element T is a norm, then we obtain a relation

$$(\Sigma a_iT^i)^2(\Sigma c_iT^i)^{-2} + (\Sigma a_iT_i)(\Sigma b_iT^i)(\Sigma c_iT^i)^{-2} + (\Sigma b_iT_i)^2(\Sigma c_iT^i)^{-2}t^{-3} = T,$$

where Σc_iT^i denotes the common denominator of the elements a, b occurring in the equation $T = \mathfrak{N}(a + bx)$. Comparing the coefficients of the different powers of T one finds

$$t^3(a_0^2 + a_0b_0) + b_0^2 = 0$$
$$a_0b_1 + a_1b_0 = c_0, \cdots.$$

In order that T be a norm the diophantine equation $b_0^2 + b_0at^3 + a_0^2t^3 = 0$ must have a solution a_0, b_0 in L. We can transform this equation to a simple form by substituting

$$r = d + cb_0 \quad \text{where} \quad d, c \in L.$$

Then

$$rb_0 = db_0 + cb_0^2 = db_0 + b_0ca_0t^3 + ca_0^2t^3$$
$$= ca_0^2t^3 + b_0(d + ca_0t^3),$$

hence r satisfies the equation

$$g(r) \equiv \begin{vmatrix} r + d & 0 \\ ca_0^2t^3 & r + d + ca_0t^3 \end{vmatrix}$$
$$= r^2 + d^2 + rca_0t^3 + dca_0t^3 + c^2a_0^2t^3$$
$$= r^2 + rca_0t^3 + (d^2 + dca_0t^3 + c^2a_0^2t^3) = 0.$$

Put $c = (a_0 t^3)^{-1}$, then

$$g(r) \equiv r^2 + r + (d^2 + d + t^{-3}) = 0.$$

The element $r = (a_0 t^3)^{-1} b_0$ satisfies the equation

$$r^2 + r + t^{-3} = 0$$

which is not solvable in L according to our general construction. But $T = \mathfrak{N}(a + bx)$ would imply that a_0, $b_0 \in L$, hence also $r \in L$ which cannot be true.

Thus

$$(T, ZL(T)/L(T)) \not\sim L(T)$$

and

$$(T, Zk/k) \not\sim k.$$

4. The structure of $\mathfrak{G}'(k)$. Whereas the structure of the group $\mathfrak{G}^{(\chi)}(k)$ is intimately related to the additive structure of the underlying field k the structure of the group $\mathfrak{G}(k)$ depends more upon the value group $\Gamma(k)$ and the multiplicative structure of the residue class field $\boldsymbol{k}$ as we shall see.

Let us assume that the field of residue classes $\boldsymbol{k}$ possesses for each integer n which is relatively prime to the characteristic χ of k a uniquely determined cyclic extension $\boldsymbol{U}_n$ of degree n; moreover, we shall suppose that $\boldsymbol{k}$ is not real in the sense of Artin-Schreier if $\chi = 0$.[15] Let $\boldsymbol{f}_n(x) \equiv x^n + \boldsymbol{a}_1 x^{n-1} + \cdots + \boldsymbol{a}_n = 0$ be a generating equation of $\boldsymbol{U}_n$, then choose elements $a_i \in k$ such that $a_i \bmod \mathfrak{p} = \boldsymbol{a}_i$. The roots of the equation $f_n(x) \equiv x^n + a_1 x^{n-1} + \cdots + a_n = 0$ generate then a cyclic unramified extension U_n of degree n over k whose field of residue classes coincides with $\boldsymbol{U}_n$.[16] Since the fields $\boldsymbol{U}_n$ are uniquely determined the unramified extensions U_n whose degrees n are relatively prime to χ are uniquely determined in a fixed algebraically closed field of k.

Let U be the infinite unramified extension of k which contains the fields U_n. Then $\Gamma(U) = \Gamma(k)$ and the field of residue classes $\boldsymbol{U}$ possesses no algebraic extensions whose degrees are relatively prime to χ. We observe that the field U is not perfect, but it is relatively perfect provided that k is relatively perfect. This remark is obvious for any equation with coefficients in U can already be considered as an equation in a sufficiently large subfield U_n of U; since U_n is relatively perfect Hensel's criterion can be applied to the equation. Moreover, we have according to Theorem 2

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[15] A field k shall be called quasi-algebraically closed if all proper and improper finite algebraic extensions of it are never centers of finite division algebras. This definition is weaker than the one usually given for quasi-algebraic closure.

[16] Cf. No. 7.

$$\mathfrak{G}(\bar{U}) = \mathfrak{G}(U) \times \bar{U}$$

where $\bar{U}$ denotes the perfect closure of U.

One shows as in the proof of Theorem 4 that the classes of algebras in $\mathfrak{G}'(k)$ which are split by U or $\bar{U}$ form a subgroup $\mathfrak{G}'_u(k)$.

THEOREM 5. *The class group of algebras $\mathfrak{G}'_u(k)$ which are split by unramified fields is isomorphic with the direct sum of λ additive groups which are isomorphic with the group of all rational numbers* mod 1 *whose denominators are relatively prime to χ.*

Proof. We verbally can repeat the proof of Theorem 4. Namely, according to the assumption concerning the unramified extensions we find that

$$\mathfrak{G}'_{U_n}(k) = \mathfrak{G}'_{U_{p_1}a_1}(k) \times \cdots \times \mathfrak{G}'_{U_{p_s}a_s}(k),$$

where $n = \prod_{i=1}^{s} p_i^{a_i}$ and $\mathfrak{G}'_{U_{p_i}{}^{a_i}}(k)$ denotes the group of all algebras which are split by $U_{p_i}{}^{a_i}$. Hence

$$\mathfrak{G}'_u(k) = \prod_{p \neq \chi}{}_x \mathfrak{G}'_{U(p)}(k)$$

where Π_x denotes the direct product of any finite number of algebras in the groups $\mathfrak{G}'_{U(p)}(k)$ consisting of all classes which can be represented by cyclic crossed products $(a, U_{p^i}/k)$. Consequently, it suffices to determine the structure of each $\mathfrak{G}'_{U(p)}(k)$. Since $U_{p^i} \subset U_{p^{i+1}} \subset \cdots \subset U_{(p)}$ we have to determine the structure of $\mathfrak{G}'_{U_{p^i}}(k)$. Let $(a, U_{p^i}/k)$ be an arbitrary algebra of the latter group. If $v(a) = \alpha = \beta + \sum_{i=1}^{\lambda} \gamma_i \theta_i$ then determine an element $b \in k$ such that $v(b) = \beta p^{-i}$ and fix elements $t_i \in k$ such that $v(t_i) = \theta_i$. Hence

$$(a, U_{p^i}/k) \sim (b^{p^i}, U_{p^i}/k) \times (\prod_{i=1}^{\lambda} t_i^{\gamma_i}, U_{p^i}/k) \times (\epsilon, U_{p^i}/k),$$

the 1st and 3rd factors of the right side are similar to k as a repetition of the argument int he proof of Theorem 4 shows.

We only have to observe that algebras $(\epsilon, U_{p^i}/k)$ whose exponent equals the degree p^i do not exist. First suppose that $\chi \neq 0$, then $k(\epsilon^{1/p^i})$ has to coincide with U_{p^i} since there can be established a one to one correspondence between the unramified extensions of k and the cyclic extensions of $\boldsymbol{k}$. Hence $\epsilon = \mathfrak{N}\epsilon^{1/p^i}$ for -1 has to be a norm. Secondly, if $\chi = 0$ then the preceding argument applies for all primes $p \neq 2$. If $p = 2$ then the existence of proper division algebras over $\boldsymbol{k}$ is excluded by the assumption that $\boldsymbol{k}$ be not real in the sense of Artin-Schreier.

Consequently,

$$\begin{aligned} k^*/\mathfrak{N}U^*_{p^i} &\cong \Gamma(k)/\mathfrak{N}\Gamma(U_{p^i}) = \Gamma(k)/p^i\Gamma(k) \\ &\cong Z_1(p^i) \times \cdots \times Z_\lambda(p^i) \end{aligned}$$

where $Z_j(p^i)$ denote cyclic groups of order p^i. Therefore $\mathfrak{G}'_{U(p)}(k)$ is isomorphic with the direct sum of λ additive groups of rational numbers mod 1 whose denominators are powers of p.

Remark. Combining Theorems 4 and 5 we see that the class groups of all algebras which are split by unramified fields is isomorphic with the direct sum of λ additive groups of all rational numbers mod 1.

LEMMA 4. *If the relatively perfect field k contains all n-th roots of unity—$(n, \chi) = 1$—then*

$$\mathfrak{G}'(\bar{U}) \cong \mathfrak{G}'(U) = \mathfrak{G}'(k) \times U \cong \mathfrak{G}'(k)/\mathfrak{G}'_u(k).$$

Proof. The proof of Lemma 2 yields that

$$k^*/k^{*n} \cong \Gamma/n\Gamma \times k^*/k^{*n} \quad \text{and} \quad U^*/U^{*n} \cong \Gamma/n\Gamma.$$

Consequently, the maximal abelian extension $K^{(n)}$ of U of exponent n is given by $K^{(n)} = U(t_1^{1/n}, \cdots, t_\lambda^{1/n}) = k(t_1^{1/n}, \cdots, t_\lambda^{1/n})U$. The ramification theory yields that any finite algebraic extension K whose degree is relatively prime to χ is contained in a suitable field $K^{(n)}$. Consequently, the class group $\mathfrak{G}'(U)$ is given by the set of subgroups $\mathfrak{G}'_{K^{(n)}}(U)$. Since the fields $K^{(n)}$ are extensions of fields over k we see that each algebra in $\mathfrak{G}'_{K^{(n)}}(U)$ can be obtained as the direct product of a suitable crossed product with the splitting field $K^{(n)}$ and of U. Hence $\mathfrak{G}'(U) = \mathfrak{G}'(k) \times U$.

LEMMA 5. *Let k be a relatively perfect field whose field of residue classes $\mathbf{k}$ possesses no algebraic extensions of degree n—$(n, \chi) = 1$—and whose value group $\Gamma(k)$ is either equal to*

$$\{P, P\chi^\infty\theta_1, \cdots, P\chi^\infty\theta_\mu\} \quad \text{or} \quad \{P, P\chi^\infty\theta_1, \cdots, P\chi^\infty\theta_\mu, \theta\Gamma^{(0)}\}$$

then

$$\mathfrak{G}'(k) = 1.$$

Proof. Suppose that $\Gamma(k) = \{P, P\chi^\infty\theta_1, \cdots, P\chi^\infty\theta_\mu\}$ then k has no algebraic extension K_n of degree n—$(n, \chi) = 1$—whatsoever, for the Galois group of any such extension has to be isomorphic with the factor group

$$\Gamma(K\ )/\Gamma(k) = \Gamma(k)/\Gamma(k).$$

Consequently $G'(k) = 1$.

If $\Gamma(k) = \{P, P\chi^\infty\theta_1, \cdots, P\chi^\infty\theta_\mu, \theta\Gamma^{(0)}\}$ then the fields $K_n = k(t^{1/n})$ where $v(t) = \theta$ are the only existing extensions of degree n. Hence any algebra of $\mathfrak{S}'(k)$ must have a suitable field K_n as splitting field. Thus we have to consider the factor group $k^*/\mathfrak{N}K^*_n$. Obviously each unit ϵ is a n-th power of a suitable unit for the equation $x^n - \epsilon \bmod \mathfrak{p} = 0$ factors completely in $\boldsymbol{k}$. Also all elements a in k whose values $v(a)$ do not involve a multiple of θ are n-th powers: choose $b \varepsilon k$ such that $v(b) = v(a)n^{-1}$ then $a = b^n \epsilon = b^n \eta^n = (b\eta)^n$. Finally, there exists an element T in K_n whose norm is equal to t; namely choose $T = t^{1/n}$ if $n \not\equiv 0$ (2) or $T = t^{1/n}\omega$ with a suitable root of unity ω if $n \equiv 0$ (2). Hence $k^* = \mathfrak{N}K^*_n$, i. e. $\mathfrak{S}'(k) = 1$.

Let k be a relatively perfect field whose value group $\Gamma(k)$ is equal to $\{P, \theta_1\Gamma^{(0)}, \cdots, \theta_\lambda\Gamma^{(0)}\}$ and whose field of residue classes $\boldsymbol{k}$ is closed with regard to extensions of degree n, $(n, \chi) = 1$. We construct the following chain of infinite abelian extensions of k:

$$\begin{aligned} K_1 &= k(\langle t_2^{1/n}, \cdots, t_\lambda^{1/n}\rangle) \\ K_2 &= k(\langle t_3^{1/n}, \cdots, t_\lambda^{1/n}\rangle), \\ K_i &= k(\langle t_{i+1}^{1/n}, \cdots, t_\lambda^{1/n}\rangle), \\ K_{\lambda-1} &= k(\langle t_\lambda^{1/n}\rangle) \end{aligned}$$

where n runs over all integers which are relatively prime to χ. We have $K_{i-1} = K_i(\langle t_i^{1/n}\rangle)$ and for the value groups

$$\begin{aligned} \Gamma(K_{i-1}) &= \{P, \theta_1\Gamma^{(0)}, \cdots, \theta_i\Gamma^{(0)}, \theta_{i+1}P\chi^\infty, \cdots, \theta_\lambda P\chi^\infty\} \\ &= \{\Gamma(K_i), \theta_i P\chi^\infty\}. \end{aligned}$$

All the fields K_i are relatively perfect.

THEOREM 6. *The group $\mathfrak{S}'(k)$ contains a series of subgroups*

$$\mathfrak{S}'(k) \supseteqq \mathfrak{S}'(k)_{K_1} \supset \cdots \supset \mathfrak{S}'(k)_{K_{i-1}} \supset \mathfrak{S}'(k)_{K_i} \supset \cdots \supset \mathfrak{S}'(k)_{K_{\lambda-1}}$$

such that

$$\begin{aligned} \mathfrak{S}'(k)_{K_{i-1}}/\mathfrak{S}'(k)_{K_i} &\cong \mathfrak{S}'(K_i)_{K_{i-1}} \\ &\cong (P\chi^\infty \bmod 1)^{(1)} + \cdots + (P\chi^\infty \bmod 1)^{(i-1)}. \end{aligned}$$

Proof. The group $\mathfrak{S}'(k)$ contains the subgroup $\mathfrak{S}'(k)_{K_i}$ which consists of all algebras which are split by K_i; therefore obviously $\mathfrak{S}'(k)_{K_{i-1}} \supset \mathfrak{S}'(k)_{K_i}$ according to the construction of the fields K_i. Evidently we have $(\mathfrak{S}'(k)_{K_{i-1}})_{K_i} = \mathfrak{S}'(k)_{K_i}$. Next we see that

$$\mathfrak{S}'(k) \times K_i = \mathfrak{S}'(K_i)$$

for k is closed with respect to algebraic extensions whose degrees are relatively prime to χ. Consequently,

$$\mathfrak{G}'(K_i) \cong \mathfrak{G}'(k)/\mathfrak{G}'(k)_{K_i}.$$

The group $\mathfrak{G}'(K_i)$ contains the group $\mathfrak{G}'(K_i)_{K_{i-1}}$ whose algebras are split by the fields $K_i(t_i^{1/n})$. Consequently the algebras

$$(t_j, K_i(t_i^{1/n})) = (t_j, k(t_i^{1/n})) \times K_i,$$

where $j = 1, \cdots, i-1$, furnish a complete set of generators of

$$\mathfrak{G}'(K_i)_{K_i(t_i^{1/n})} \subset \mathfrak{G}'(K_i)_{K_{i-1}}.$$

Next we have

$$\begin{aligned}\mathfrak{G}'(k)_{K_{i-1}} \times K_i &= \mathfrak{G}'(K_i)_{K_{i-1}} = \mathfrak{G}'(k)_{K_{i-1}}/(\mathfrak{G}'(k)_{K_{i-1}})_{K_i}\\ &\cong \mathfrak{G}'(k)_{K_{i-1}}/\mathfrak{G}'(k)_{K_i} \cong \mathfrak{G}'(K_i)_{K_{i-1}}\\ &\cong (\mathrm{P}\chi^\infty \bmod 1)^{(1)} + \cdots + (\mathrm{P}\chi^\infty \bmod 1)^{(i-1)}.\end{aligned}$$

The chain of subgroups has to start with $\mathfrak{G}'(k)_{K_1}$ according to Lemma 5.

LEMMA 6. *If $(a_{\sigma,\tau}, K/k)$ and $(b_{\Sigma,\mathrm{T}}, N/k)$ are two crossed products such that $K \cap N = k$ then $(a_{\sigma,\tau}, K/k) \times (b_{\Sigma,\mathrm{T}}, N/k) \sim (c_{S,T}, KN/k)$ where $\| c_{S,T} \| = \| a_{\sigma,\tau} \| \otimes \| b_{\Sigma,\mathrm{T}} \|$ is the Kronecker product of the two matrices $\| a_{\sigma,\tau} \|$, $\| b_{\Sigma,\mathrm{T}} \|$ which represent the factor sets.*

Proof. The algebra $(a_{\sigma,\tau}, K/k)$ has also the field KN as normal splitting field. Hence $(a_{\sigma,\tau}, K/k) \sim (a'_{S,T}, KN/k)$ where $a'_{S,T} = a'_{S',T}$, for $S \equiv S'(\{\Sigma\})$, $T \equiv T'(\{\Sigma\})$ and $a'_{\sigma\{\Sigma\},\tau\{\Sigma\}} = a_{\sigma,\tau}$ since the Galois group of KN/k is according to our assumption equal to the direct product of the Galois groups $\{\sigma\}$ and $\{\Sigma\}$ of K and N respectively.[17] The matrix $\| a'_{S,T} \|$ of degree $[KN:k]^2 = [K:k]^2[N:k]^2 = n^2m^2$ is equal to the Kronecker product of the matrix $\| a_{\sigma,\tau} \|$ and the unit matrix of degree $[N:k]^2$ as the relation $a'_{\sigma\{\Sigma\},\tau\{\Sigma\}} = a_{\sigma,\tau}$ implies. Hence $(a_{\sigma,\tau}, K/k) \sim (\| a_{\sigma,\tau} \| \otimes E_{m^2}, KN/k)$.
Similarly we obtain

$$(b_{\Sigma,\mathrm{T}}, N/k) \sim (E_{n^2} \otimes \| b_{\Sigma,\mathrm{T}} \|, KN/k)$$

when we take in account a fixed ordering of the elements of $\{\sigma\} \times \{\Sigma\}$ with regard to the direct factors. Combining the two representations we get

$$(a_{\sigma,\tau}, K/k) \times (b_{\Sigma,\mathrm{T}}, N/k) \sim (c_{S,T}, KN/k) = (\| a_{\sigma,\tau} \| \otimes \| b_{\Sigma,\mathrm{T}} \|, KN/k).$$

[17] D. Chap. V, § 4.

Let $\mathfrak{A}$ be an arbitrary simple algebra of the group $\mathfrak{S}'(k)$ whose degree is relatively prime to χ. Then every maximally commutative subfield K of $\mathfrak{A}$ is abelian. Let $K = Z_1 \cdots Z_\mu$ be an arbitrary but fixed representation of K as the join of relatively prime cyclic fields Z_i with the respective Galois groups $\{\sigma_i\}$. The algebra $\mathfrak{A}$ has the form

$$(a_{S,T}, K/k)$$

where $S, T, \cdots$ denote the elements of the Galois group $\{\sigma_1\} \times \cdots \times \{\sigma_\mu\}$ of K over k. We shall call components $a_{S,T}$ of the factor set *side components* if $S = \sigma_i^{s_i} \neq \epsilon$, $T = \sigma_j^{t_j} \neq \epsilon$ are not elements of the same subgroup $\{\sigma_i\}$.

We then can prove a general theorem which characterizes the classes of $\mathfrak{S}'(k)$ which are similar to direct products of classes which can be represented by cyclic algebras.

THEOREM 7. *A class of $\mathfrak{S}'(k)$ is similar to the direct product cyclic classes if and only if the given class possesses an abelian splitting field $K = Z_1, \cdots, Z_\mu$ such that the side components of a suitable associated factor set belonging to the class are equal to unity.*

Proof. Suppose that the algebra $\mathfrak{A}$ of the given class in $\mathfrak{S}'(k)$ possesses a splitting field K such that for a suitable representation of K as the join of cyclic fields Z_i there does exist a factor set $a_{S,T}$ whose side components (with respect to the fixed decomposition of K) are equal to unity. Then

$$u_{\sigma_i\nu_i} u_{\sigma_j\nu_j} = u_{\sigma_i{}^{\nu_i}\sigma_j{}^{\nu_j}} = u_{\sigma_j{}^{\nu_j}\sigma_i{}^{\nu_i}}$$

and

$$u^{-1}_{\sigma_i\nu_i} z^{(j)} u_{\sigma_i\nu_i} = z^{(j)}$$

for any pair (i, j) of different indices, $z^{(j)} \varepsilon Z_j$. If $[K:k] = n = n_1 \cdots n_\mu = [Z_1:k] \cdots [Z_\mu:k]$, $[\mathfrak{A}:k] = n^2$ then all sums

$$\sum_{l=1}^{n_i} z_l{}^{(i)} u_{\sigma_i}{}^{l}$$

constitute a normal simple algebra $\mathfrak{A}^{(i)}$ of degree $n^{(i)}$, $i = 1, 2, \cdots, \mu$. The algebras $\mathfrak{A}^{(i)}$ are contained in $\mathfrak{A}$ and we can form the direct product $\mathfrak{A}^{(1)} \times \cdots \times \mathfrak{A}^{(\mu)}$ within $\mathfrak{A}$. Namely, the elements of different algebras $\mathfrak{A}^{(i)}$, $\mathfrak{A}^{(j)}$ permute with each other as a consequence of our assumption on the side components, and $[\mathfrak{A}:k] = \prod_{i=1}^{\mu} [\mathfrak{A}_i:k]$. Now we can normalize the factor set belonging to each component $\mathfrak{A}_i$ to the cyclic normal form

$$\mathfrak{A}_i = (a_i, Z^{(i)}/k)$$

where $a_i \varepsilon k$. Since all units ϵ of k are n_i-th powers $(i = 1, \cdots, \mu)$ we can write

$$\mathfrak{A}_i = (\prod_{j=1}^{\lambda} t_j^{\nu_{ij}}, Z^{(i)}/k).$$

Moreover, it follows if we apply Lemma 5 that the original factor set $a_{S,T}$ is associated to a factor set which lies in the ground field k. However, $a_{S,T} \varepsilon k$ does not imply that A is similar to the direct product of cyclic algebras.

Conversely, if an algebra $\mathfrak{A}$ is similar to the direct product of cyclic algebras $(a_i, Z^{(i)}/k)$ when the splitting fields $Z^{(i)}$ are relatively prime over k $(i = 1, 2, \cdots, \mu)$ then $\mathfrak{A}$ is similar to an algebra which has the join $K = Z_1 \cdots Z_\mu$ as splitting field such that the side components of the corresponding factor set are equal to unity. Namely, each component $(a_i, Z^{(i)}/k)$ has the following matrix representing the complete factor set

$$\left\| \begin{matrix} 1 & \cdot & \cdot & \cdots & \cdot & \cdot & 1 \\ \cdot & & & & & \cdot & \cdot \\ \cdot & & & & \cdot & & a_i \\ \vdots & & & \iddots & & \iddots & \vdots \\ \cdot & & \iddots & & \iddots & & \cdot \\ \cdot & \cdot & & \cdot & & & \cdot \\ 1 & a_i & \cdot & \cdots & \cdot & \cdot & a_i \end{matrix} \right\| = A_i$$

of degree $n_i = [Z_i : k]$.[18] Hence

$$\begin{aligned} \mathfrak{A} &\sim (a_1, Z^{(1)}/k) \times \cdots \times (a_\mu, Z^{(\mu)}/k) \\ &\sim (A_1 \otimes \cdots \otimes A_\mu, K/k) \end{aligned}$$

according to Lemma 6. The side components of $A_1 \otimes \cdots \otimes A_\mu$ obviously are equal to unity according to the structure of Kronecker products.

Let $\mathfrak{D}$ be an arbitrary normal division algebra over the relatively perfect field k, such that its degree n is relatively prime to the characteristic χ. Then the valuation $\mathfrak{p}$ of k can be extended to $\mathfrak{D}$ by defining the value $v(a) \varepsilon \Gamma(\mathfrak{D})$ of any element $a \neq 0$ in $\mathfrak{D}$ to be equal to

$$\frac{1}{n} v(\mathfrak{N}_{\mathrm{red}}\ a)$$

where $\mathfrak{N}_{\mathrm{red}}$ denotes the reduced norm of $\mathfrak{D}$ over k. Since $\Gamma(k)$ is assumed to have a finite irrational base $\theta_1, \cdots, \theta_\lambda$ we find that the factor group

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[18] O. F. G. Schilling, "The structure of certain rational infinite algebras," *Duke Mathematical Journal*, vol. 3 (1937).

$$\Gamma(\mathfrak{D})/\Gamma(k)$$

is a finite abelian group whose invariants are proper or improper divisors of the degree n. Obviously, $[\Gamma(\mathfrak{D})/\Gamma(k):1] \leqq [\mathfrak{D}:k]$ for the representatives of the factor groups are values of elements which are linearly independent over k.

THEOREM 8. *A division normal algebra $\mathfrak{D}$ of degree n over k is equal to a cyclic algebra if and only if the factor group $\Gamma(\mathfrak{D})/\Gamma(k)$ contains at least one element of exact order n.*

Proof. Let a be an element of $\mathfrak{D}$ such that exactly $nv(a)$ lies in $\Gamma(k)$. Hence $v(a) = \alpha + \sum_{i=1}^{\lambda} \gamma_i \theta_i n^{-1}$ where at least one integer γ_i is relatively prime to n, $\alpha \varepsilon P$. Consequently, the field $k(a)$ is a maximally commutative subfield of $\mathfrak{D}$. It must be a cyclic extension, for

$$k(a) = k(b_1^{1/n_1}, \cdots, b_s^{1/n_s}), \qquad s > 1,$$

$b_i \varepsilon k$, $n_1 \cdots n_s = n$, $n_i /\!\!/ n$ would imply that $\Gamma(k(a))/\Gamma(k)$ has the invariants $(n_1, n_2, \cdots, n_s)$ whereas a is an element which has the exact order n. Therefore

$$\mathfrak{D} = (c, k(a)) = (c, k(b^{1/n}))$$

with a suitable quantity $c \neq 0$.

Conversely, if $\mathfrak{D} = (c, k(b^{1/n}))$ then

$$v(b^{1/n}) = \frac{1}{n} v(b) \, \varepsilon \, \Gamma(\mathfrak{D})$$

is an element of $\Gamma(\mathfrak{D})$ which has the exact order n with respect to $\Gamma(k)$.

THEOREM 9. *A crossed product $(a_{\sigma,\tau}, K/k)$ is never a division algebra if there exists a subset a_{ρ_i,ρ_j} of $a_{\sigma,\tau}$ which belongs to a cyclic subgroup $\{\rho\}$ of the Galois group $\{\sigma, \tau, \cdots\}$ and which has the form $b_{\rho_i\rho_j} \varepsilon \rho_i, \rho_j$ where $v(b_{\rho_i,\rho_j}) \varepsilon$ P.*

Proof. Let K_ρ be the subfield of K which belongs to the subgroup $\{\rho\}$ of $\{\sigma, \tau, \cdots\}$. Then

$$(a_{\sigma,\tau}, K/k) \times K_\rho \sim (a_{\rho_i,\rho_j}, K/K_\rho) = (\prod_{i=1}^{m} a_{\rho_i,\rho_j}, K/K_\rho)$$

if $[K:K_\rho] = m$.

The element $\prod_{i=1}^{m} a_{\rho^i,\rho} \,\varepsilon\, K_\rho$ has a rational value β according to assumption.

Choose now an element $b \,\varepsilon\, K_\rho$ whose value is equal to β/m, then

$$\prod_{i=1}^{m} a_{\rho^i,\rho} b^{-m}$$

is a unit ϵ_ρ of K_ρ and consequently

$$\begin{aligned}(\prod_{i=1}^{m} a_{\rho^i,\rho}, K_\rho/K_\rho) &\sim (b^m, K/K_\rho)(\epsilon_\rho, K/K_\rho)\\ &\sim (\epsilon_\rho, K/K_\rho)\\ &\sim K_\rho\end{aligned}$$

since each unit ϵ_ρ is a m-th power, or

$$(a_{\sigma,\tau}, K/k) \times K_\rho \sim K_\rho.$$

Suppose now that $(a_{\sigma,\tau}, K/k)$ would be a division algebra, then the degree of each of its splitting fields would be a multiple of $[K:k]$. We just showed that the field K_ρ of degree $[K:k]m^{-1} < [K:k]$ is a splitting field, but this contradicts our assumption that $(a_{\sigma,\tau}, K/k)$ were a division algebra.

We wish to conclude this section by some general remarks about the structure of normal simple algebras over a relatively perfect field k whose field of residue classes $\boldsymbol{k}$ is an arbitrary field of characteristic χ.

We have seen that there do exist normal simple algebras $\mathfrak{A}$ over k which are not split by any unramified extension U of k provided that $\lambda \geqq 2$. Thus we term a division algebra $\mathfrak{D}$ over k to be *totally ramified* if its algebra of residue classes $\boldsymbol{D}$ coincides with the field $\boldsymbol{k}$. A division algebra $\mathfrak{D}$ is called *unramified* if its algebra of residue classes $\boldsymbol{D}$ has the same degree over $\boldsymbol{k}$ as $\mathfrak{D}$ over k. It is a known fact that unramified algebras exist if and only if $\boldsymbol{k}$ is not quasi-algebraically closed.[19] As in the theory of fields which are perfect with respect to a discrete valuation we can prove the following existence theorem.[20]

THEOREM 10. *If $\boldsymbol{D}$ is a normal division algebra over $\boldsymbol{k}$ then there exists at least one unramified division algebra $\mathfrak{D}$ over k whose algebra of residue classes is equal to $\boldsymbol{D}$.*

THEOREM 11. *Each algebra of $\mathfrak{S}_u(k)$ is similar to the direct product of cyclic algebras and of an unramified division algebra.*

Proof. Let $\mathfrak{A} = (a_{\sigma,\tau}, U/k)$ be an arbitrary algebra of $\mathfrak{S}_u(k)$, then $v(a_{\sigma,\tau}) = \alpha(\sigma,\tau) + \sum_{i=1}^{\lambda} \gamma_i(\sigma,\tau)\theta_i$ where $\alpha(\sigma,\tau) \in \mathrm{P}$, $\gamma_i(\sigma,\tau) \in \Gamma^{(0)}$ since $\Gamma(U) = \Gamma(k)$. Determine elements $b_{\sigma,\tau} \in k$ such that

$$v(b_{\sigma,\tau}) = [U:k]^{-1}\alpha(\sigma,\tau)$$

[19] See R. Brauer, "Über die Konstruktion der Schiefkörper, die von endlichen Rang in Bezug auf ein gegebenes Zentrum sind," § 7, Theorem 12, *Crelle*, vol. 168 (1932).

[20] See T. Nakayama, "Divisionsalgebren über diskret bewerkten perfekten Körpern," *Crelle*, vol. 177 (1937).

and fix elements t_i such that $v(t_i) = \theta_i$. Then

$$a_{\sigma,\tau} b_{\sigma,\tau} \prod_{i=1}^{\lambda} t_i^{-\gamma_i(\sigma,\tau)}$$

is a factor set consisting of units $\epsilon_{\sigma,\tau} \varepsilon U$ for $b_{\sigma,\tau} \varepsilon k$ and

$$\gamma_i(\sigma, \tau\rho) + \gamma_i(\tau, \rho) = \gamma_i(\sigma\tau, \rho) + \gamma_i(\sigma, \tau) \qquad (i = 1, 2, \cdots, \lambda).$$

Hence

$$\begin{aligned}\mathfrak{A} &\sim (\epsilon_{\sigma,\tau}, U/k)(b_{\sigma,\tau}^{-1}, U/k)(\prod_{i=1}^{\lambda} t_i^{\gamma_i(\sigma,\tau)}, U/k) \\ &\sim (t_1^{\gamma_1(\sigma,\tau)}, U/k) \times \cdots \times (t_\lambda^{\gamma_\lambda(\sigma,\tau)}, U/k)(\epsilon_{\sigma,\tau}, U/k) \\ &\sim (t_1, Z_1/k) \times \cdots \times (t_\lambda, Z_\lambda/k)(\epsilon_{\sigma,\tau}, U/k)\end{aligned}$$

where Z_i denote the cyclic subfields of U which are induced by the characters related to the sets of addends $\gamma_i(\sigma, \tau)$. The algebras $(t_i, Z_i/k)$ whose generating automorphisms are normalized are division algebras whose algebras of residue classes are equal to $\mathbf{Z}_i$. The algebra $(\epsilon_{\sigma,\tau}, U/k)$ is obviously similar to an unramified division algebra.

If we suppose that all unramified are products of cyclic algebras—which amounts to assuming that all algebras over $\boldsymbol{k}$ are direct products of cyclic algebras—then each algebra of $\mathfrak{G}_u(k)$ is representable as the direct product of cyclic algebras. In general, however, $\mathfrak{G}_u(k)$ need not exhaust the group of all classes of algebras which can be represented as direct products of cyclic algebras. Namely, if $\lambda = 4$ then there exists a division algebra of degree 4 which is totally ramified:

$$(t_1, k(\sqrt{t_2})/k)(t_3, k(\sqrt{t_4})/k).$$

All classes which can be represented by unramified division algebras evidently form a group. The classes which can be represented by completely ramified division algebras do not form a group. Consider the following example. Assume that k contains the n-th roots of unity and that there does exist a unit ϵ in k which is not the norm of an element of the cyclic extension $k(t_2^{1/n})$ $(\lambda = 2)$. The two algebras

$$\mathfrak{A}_1 = (t_2, k(t_1^{1/n})/k) \quad \text{and} \quad \mathfrak{A}_2 = (t_2, k((\epsilon t_1)^{1/n})/k)$$

obviously are division algebras. They are similar to the algebras

$$(t_1^{-1}, k(t_2^{1/n})/k) \quad \text{and} \quad (t_1^{-1}\epsilon^{-1}, k(t_2^{1/n})/k)$$

respectively. Then

$$\mathfrak{A}_1^{-1} \times \mathfrak{A}_2 \sim (\epsilon^{-1}, k(t_2^{1/n})/k) \sim (t_2, k(\epsilon^{1/n})/k),$$

the resulting algebra is a division algebra whose residue algebra is equal to the field of residue classes belonging to the unramified cyclic extension $k(\epsilon^{1/n})$ of k.

It is easy to see that in case of $\lambda = 0$ there do exist only unramified algebras in the class group $\mathfrak{S}'(k)$. Then each algebra $\mathfrak{A}$ in $\mathfrak{S}'(k)$ has an unramified normal splitting field U,

$$\mathfrak{A} = (a_{\sigma,\tau}, U/k)$$

and

$$\mathfrak{A} \sim (\epsilon_{\sigma,\tau}, U/k)$$

where

$$\epsilon_{\sigma,\tau} = a_{\sigma,\tau} b_{\sigma,\tau}$$

when

$$v(b_{\sigma,\tau}) = [U:k]^{-1} v(a_{\sigma,\tau}).$$

It is also possible to develop the arithmetic theory of simple algebras $\mathfrak{A} = \mathfrak{D}_r$. However, in the proofs one has to avoid any argument which may necessitate the use of bases of orders and ideals. The existence of at least one maximal order $\mathfrak{M}$ which contains a given order of highest rank in $\mathfrak{A}$ can be proved by applying well ordering. Since $\Gamma(k)$ is everywhere dense in the additive group of all real numbers it can be shown that an algebra $\mathfrak{A} = \mathfrak{D}_r$ $(r > 1)$ contains continuously many maximal orders. We do not wish to develop the arithmetic theory here for it has only indirect bearing upon the problem of local class field theory.

5. Generalization of classical local class field theory. We showed in section 3 by two examples that there exist under certain circumstances proper division algebras in $\mathfrak{S}'(k)$ if the field of residue classes $\boldsymbol{k}$ is algebraically closed, although $\lambda = 0, 1$. We can expect for this reason a satisfactory and complete characterization of the abelian fields K over k whose degrees are relatively prime to the characteristic χ. Theorems 5, 6, and 7 of the last section show that the number λ of rationally independent irrational generators of the value group $\Gamma(k)$—they may be ordered according to increasing absolute values—can assume the following determinations in order that the group $\mathfrak{S}'(k)$ be isomorphic with the additive group P_{χ^∞} mod 1:

Case i). For each n there exists a uniquely determined extension $\boldsymbol{U}_n$ of degree n over $\boldsymbol{k}$ then $\lambda = 1$, and

Case ii). The field of residue classes $\boldsymbol{k}$ is algebraically closed then $\lambda = 2$.

We shall put more stress on the first case since one readily can extend the methods used there to the second case. First of all, Lemma 5 shows that

the group $\mathfrak{G}'(U)$ which belongs to the unramified closure of the given ground fields is equal to one. Since $\mathfrak{G}'(k) \times U \subseteq \mathfrak{G}'(U)$ we see that all classes of algebras in $\mathfrak{G}'(k)$ possess unramified cyclic splitting fields. Therefore $\mathfrak{G}'(k) = \mathfrak{G}'_u(k)$ and hence

$$\mathfrak{G}'(k) \cong \mathrm{P}_\chi{}^\infty \bmod 1$$

according to Theorem 5. Thus we proved

THEOREM 12. *The class group of algebras $\mathfrak{G}'(k)$ over a relatively perfect field whose value group $\Gamma(k) = \{\mathrm{P}, \theta_1\Gamma^{(0)}\}$ and which possesses for each integer n —$(n, \chi) = 1$ —a unique unramified extension of degree n, is isomorphic to the additive group $\mathrm{P}_\chi{}^\infty$ mod 1.*

In the following statements it shall be understood that the relative degree of any field K and the degree of any division algebra considered, shall be relatively prime to the characteristic χ.

THEOREM 13. *The Galois group of any abelian extension K over k is isomorphic with the norm class group $k^*/\mathfrak{N}K^*$.*

Proof. We first show that each field K of degree n is splitting field of each normal division algebra $\mathfrak{D}$ of degree n. Let $\mathfrak{D}$ be an arbitrary division algebra of degree n then according to Theorem 12

$$\mathfrak{D} = (t^\nu, U_n/k)$$

where $(\nu, n) = 1$ and $v(t) = \theta$, for a fixed choice of the generating automorphism of the Galois group of U_n with respect to k.

The general ramification theory of A. Ostrowski yields in the particular case we have to consider, that the inertial field $K(I)$ of the given field K is a field U_f, where f/n. Moreover, $[K:K(I)] = n/f = e$ and hence as a consequence of our assumption on $\Gamma(k)$

$$t = T^e\mathrm{E}$$

where T denotes an element of K such that $v(\mathfrak{N}T) = \theta$, E a suitable unit of K.

Consider now the algebra [21]

$$\begin{aligned} \mathfrak{D} \times K &\sim (T^e\mathrm{E}, U_nK/K) \\ &\sim (T^e, U_nK/K)(\mathrm{E}, U_nK/K) \\ &\sim (T, U_nK/K)^e(\mathrm{E}, U_nK/K) \\ &\sim (\mathrm{E}, U_nK/K) \\ &\sim K \end{aligned}$$

[21] Compare with C. Chevalley, "La théorie du symbole de restes normiques," *Crelle*, vol. 169 (1933).

where $(T^e, U_nK/K) \sim K$ as a consequence of $[U_nK:K] = e$ and where $(\mathrm{E}, U_nK/K) \sim K$ since the fields of residue classes $\boldsymbol{k}$ and a fortiori $\boldsymbol{K}$ are quasi-algebraically closed. Now let Z_n be an arbitrary cyclic extension of degree n then

$$G'(k)_{Z_n} = G'(k)_{U_n} \cong \mathrm{Z}(n)$$

for

$$\mathfrak{D} \times Z_n \sim Z_n$$

where $\mathfrak{D}$ is any generator of the cyclic class group $\mathfrak{G}'(k)_{U_n} \subset \mathrm{P}_\chi{}^\infty \bmod 1$; $\mathrm{Z}(n)$ denotes a cyclic group of order n. Consequently,

$$k^*/\mathfrak{N} Z^*_n \cong \mathrm{Z}(n)$$

holds for any cyclic field Z_n. Moreover, the same result can be proved for the cyclic fields over arbitrary finite extensions L of k since the residue fields $\boldsymbol{L}$ are still quasi-algebraically closed and possess for each n exactly one unramified extension of degree n.

Next let K be an arbitrary normal extension of degree n over k. The ramification theory yields immediately that the Galois group of K over k is solvable. Using a composition chain of the Galois group such that the respective factor groups are cyclic we find that we can find a chain of subfields $k \subset \cdots \subset K_{i-1} \subset K_i \subset \cdots \subset K$ between k and K such that the fields K_i are cyclic extensions of the fields K_{i-1}. Application of the previously obtained result concerning the norm factor groups of cyclic extensions yields

$$K^*_{i-1}/\mathfrak{N}_i K^*_i \cong \mathrm{Z}([K_i : K_{i-1}])$$

when $\mathfrak{N}_i$ denotes the relative norm taken from K_i to K_{i-1}. Consequently

$$[k^* : \mathfrak{N} K^*] \leqq [K : k]$$

since the norms are transitive multiplicative functions. Now let K be an abelian extension of degree n with the Galois group $\Sigma = \{\sigma, \tau, \cdots\}$. Then there exists at least one factor set $a_{\sigma,\tau}$ of elements in K such that the n-th and no earlier power of it is equivalent to the unit factor set. The existence of $a_{\sigma,\tau}$ is the immediate consequence of the similarity relation

$$\mathfrak{D} \times K \sim K$$

where $\mathfrak{D}$ denotes an arbitrary generating algebra of $\mathfrak{G}'(k)_{U_n}$. We form the elements

$$f_\sigma = \prod_{\tau \in \Sigma} a_{\sigma,\tau},$$

they determine an isomorphism between Σ and a subgroup of $k^*/\mathfrak{N}K^*$.[22] Since $[k^*:\mathfrak{N}K^*] \leqq n$ we have

$$\Sigma \cong k^*/\mathfrak{N}K^*.$$

Thus we proved the isomorphism theorem for the abelian extensions of k.

Remark. The group of *all* algebras which have unramified splitting fields, i.e. $\mathfrak{S}'_u(k)$ is isomorphic with the additive group of all rational fractions mod 1 provided that also for the degrees χ^m there exist uniquely determined cyclic unramified fields.

In the classical theory the isomorphism $\Sigma \cong k^*/\mathfrak{N}K^*$ is given by a definite realization by the norm residue symbol.[23] The possibility of normalizing the aforementioned isomorphism depends upon the fact that an arithmetically significant automorphism which generates the cyclic Galois group belonging to the unramified fields can be found. Thus, if we make no special assumption on the algebraic structure of the residue class field $\boldsymbol{k}$ we have to find a substitute for the so-called Frobenius automorphism.

Let U be the infinite cyclic unramified extension of k which is equal to the join of all unramified extensions U_n of k. The Galois group of U with regard to k is then a totally disconnected compact group $O = \{o, \cdots\}$.[24] It is easy to give an explicit representation of the group O. Let then

$$k \subset U^{(1)} \subset U^{(2)} \subset \cdots \subset U^{(i-1)} \subset U^{(i)} \subset \cdots \subset U$$

be an arbitrary—but necessarily enumerable—tower of unramified fields approximating U and let $O_i = \{o_i, \cdots\}$ be the Galois groups of $U^{(i)}$ with respect to k. Since $U^{(i)}/U^{(i-1)}$ and $U^{(i)}/k$ are normal the substitutions o_i of O_i induce exactly the substitutions o_{i-1} of O_{i-1}, or O_{i-1} is a homomorphic map $O_i * \phi_i$ of O_i. Then O can be isomorphically represented by the sequences $[o_1, o_2, \cdots, o_{i-1}, o_i, \cdots]$ of elements $o_i \,\varepsilon\, O_i$ such that $o_{i-1} = o_i * \phi_i$.

In order to obtain a substitute for the Frobenius substitution let us fix an arbitrary approximation of U and an arbitrary generating automorphism o_1 of U_1. Then we take an arbitrary but fixed element ψ of our uniquely determined representation of O which induces the substitution o_1 in U_1. Since each unramified field U_n is contained in U the automorphism ψ_n determines

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[22] Yasuo Akizuki, "Eine homomorphe Zuordnung der Elemente der galoisschen Gruppe zu den Elementen einer Untergruppe der Normenklassengruppe," *Mathematische Annalen*, vol. 112 (1936).

[23] Cf. paper quoted under No. 19.

[24] J. Herbrand, "Théorie arithmétique des corps de nombres de degré infini, II. Extensions algébriques de degré infini," *Mathematische Annalen*, vol. 108 (1933).

uniquely a *generating* substitution ψ_n of the Galois group belonging to U_n over k. We shall use this element ψ_n in order to normalize the representation of the elements of the class group $G'(k)_{U_n}$. According to the theory of cyclic crossed products we know that any cyclic algebra (a, U_n, σ_n) is equal to (a^r, U_n, ψ_n) where $\sigma_n = \psi_n{}^s$ is a generating substitution of the Galois group and $sr \equiv 1 \pmod{n}$. Hence each algebra of $G'(k)_{U_n}$ is similar to a power of the normalized division algebra (t, U_n, ψ_n) where t denotes a fixed element of k whose value is equal to $\theta \varepsilon \Gamma(k)$. It is readily seen that this normalization for a fixed degree n is consistent with the normalization of all other degrees m which are multiples of n; one only has to take in account the familiar rules concerning cyclic crossed products. The substitutions ψ_n and normalizations (t, U_n, ψ_n) have *algebraically* the same significance as the classical normalizations with respect to the Frobenius substitution. Consequently, we are able to define a generalized *norm residue symbol.* Let (a, Z_n, S_n) be an algebra which belongs to the class group of algebras split by the arbitrary but fixed cyclic field Z_n; S_n a generating automorphism of the Galois group. Suppose now that

$$(a, Z_n, S_n) \sim (t, U_n, \psi_n)^a,$$

then we define the norm residue symbol (a, Z_n) to be the substitution $S_n{}^a$. Using this definition we verbally can take over the theorems of the classical local class field theory. One also observes going over the known proof of the existence, ordering and translation theorems that the latter are preserved for the type of relatively perfect fields we consider under case one. Thus we can state

THEOREM 14. *Let k be a relatively perfect field whose value group $\Gamma(k)$ is isomorphic with $\{\mathrm{P}, \theta\Gamma^{(0)}\}$ and whose residue class field $\mathbf{k}$ admits for each n exactly one cyclic extension of degree n, $(n, \chi) = 1$. Then the law of reciprocity, existence, ordering and translation theorems of the classical local class field theory hold for k as ground field.*

In case two where the field of residue classes $\boldsymbol{k}$ is algebraically closed and where $\lambda = 2$ we can repeat all preceding considerations. However, it is necessary to consider either $k(\langle t_1{}^{1/n}\rangle)$ or $k(\langle t_2{}^{1/n}\rangle)$ as the universal splitting field of the group $\mathfrak{G}'(k)$. In either case we obtain the equivalent of the local class field theory. It only has to be pointed out that the generalization of the Frobenius automorphism has no more an invariantive arithmetical significance.

Remark. It is possible to establish the results of classical local class field theory even if the assumptions made on page 695 are slightly weakened. Namely,

we can generalize the value group Γ to arbitrary subgroups of the additive group of all real numbers with the restriction that

$$\Gamma/n\Gamma \cong \frac{1}{n}\Gamma/\Gamma \cong Z(n) \text{ for all } n \text{ in case i)},$$

and

$$\Gamma/n\Gamma \cong \frac{1}{n}\Gamma/\Gamma \cong Z(n) \times Z(n) \text{ for all } n \text{ in case ii)}.$$

However, it should be observed that the representatives of the factor groups can well be different for different values of n. Nevertheless, for powers of a single prime one can choose one and the same representatives, if the integer n is composite then the representatives can be built up additively from the representatives of the prime factors of n. In generalizing the proofs of the preceding theorems this fact has to be taken in account, in particular with regard to the selection of elements of the perfect fields whose values are equal to representatives of certain factor groups. Going over the proofs of the preceding theorems one easily observes that the generalization is obvious.

6. Examples of perfect fields with prescribed value group. We wish to give in this section several examples of perfect fields k whose value groups $\Gamma(k)$ are isomorphic to a prescribed group $\{P, \theta_1\Gamma^{(0)}, \cdots, \theta_\lambda\Gamma^{(0)}\}$ of real numbers and whose fields of residue classes $\boldsymbol{k}$ are given too.

Let $\boldsymbol{k}$ be an arbitrary field of characteristic χ. Consider then the field of rational functions $\boldsymbol{k}(x_0, x_1, \cdots, x_\lambda)$ and put

$$(*) \qquad \begin{aligned} v(x_0) &= 1 \\ v(x_i) &= \theta_i \qquad (i = 1, \cdots, \lambda) \end{aligned}$$

where θ_i are given real numbers which are rationally independent. The set of $\lambda + 1$ relations (*) defines then a valuation $\mathfrak{p}$ on $\boldsymbol{k}(x_0, x_1, \cdots, x_\lambda)$ whose valuation ring contains the ring of polynomials $\boldsymbol{k}[x_0, x_1, \cdots, x_\lambda]$ and whose field of residue classes obviously coincides with the given field $\boldsymbol{k}$. Next we adjoin to $\boldsymbol{k}(x_0, x_1, \cdots, x_\lambda)$ the radicals x_0^{1/p^j} where p runs over all primes which are different from the characteristic χ and where $j = 1, 2, \cdots$. The resulting field $\boldsymbol{k}(x_0, x_1, \cdots, x_\lambda)(\langle x_0^{1/p^j}\rangle)$ admits then, according to the general valuation theory, a valuation whose value group is equal to

$$\{P_\chi^\infty, \theta_1\Gamma^{(0)}, \cdots, \theta_\lambda\Gamma^{(0)}\}$$

and its field of residue classes coincides with $\boldsymbol{k}$.

Now we distinguish two cases

$$\text{(i)} \quad \chi = 0 \qquad\qquad \text{(ii)} \quad \chi \neq 0$$

Case i). We adjoined all n-th roots of x_0 and thus obtained an infinite algebraic extension K of $\boldsymbol{k}(x_0, x_1, \cdots, x_\lambda)(\langle x_0^{1/n}\rangle)$ on which there is defined a valuation $\mathfrak{P}$ whose value group is equal to $\{P; \theta_1\Gamma^{(0)}, \cdots, \theta_\lambda\Gamma^{(0)}\}$ and whose field of residue classes is still equal to $\boldsymbol{k}$. Taking the $\mathfrak{P}$-adic closure of K we obtain a perfect field k which has the desired properties.

Case ii). We adjoin to $\boldsymbol{k}(x_0, x_1, \cdots, x_\lambda)(\langle x_0^{1/p^j}\rangle)$ the roots of the cyclic equations

$$y_\nu^\chi \dot{-} y_\nu = a_\nu \text{ where } a_\nu = (1/x_0, \cdots, 1/x_0) \text{ is a vector of } \nu \text{ elements,} \quad (\nu = 1, 2, \cdots).$$

The valuation $\mathfrak{p}$ of $\boldsymbol{k}(x_0, x_1, \cdots, x_\lambda)$ possesses again at least one extension $\mathfrak{P}$ in the infinite algebraic extension

$$K = \boldsymbol{k}(x_0, x_1, \cdots, x_\lambda)(\langle x_0^{1/p^j}\rangle)(\langle y_\nu\rangle).$$

The $\mathfrak{P}$-adic closure k of K is then a field which possesses the required value group and residue field.

It is also possible to prove the existence of fields k with given value group by an entirely different method. Let now $\boldsymbol{k}$ be an algebraically closed field of characteristic χ. Consider the field

$$\boldsymbol{k}(x_0, y; x_1, \cdots, x_\lambda)$$

where

$$y = \sum_{i=0}^{\infty} \boldsymbol{a}_i x_0^{r_i}$$

and where $\{r_i\} \to \infty$ is a sequence of rational numbers whose denominators are powers of primes $p \neq \chi$, so that ultimately arbitrarily high powers of all the primes p occur in the denominators. Obviously, infinitely many such selections can be made for the sequence $\{r_i\}$. Put then

$$\begin{aligned} v(x_0) &= 1 \\ v(y) &= r_0 \\ v(x_i) &= \theta_i \qquad (i = 1, \cdots, \lambda). \end{aligned}$$

We thus determine a valuation of the field $L = \boldsymbol{k}(x_0, y; x_1, \cdots, x_\lambda)$ whose field of residue classes is equal to $\boldsymbol{k}$. The value group belonging to $\mathfrak{p}$ contains the components

$$\Gamma^{(0)} \leqq r_0\Gamma^{(0)}, \theta_1\Gamma^{(0)}, \cdots, \theta_\lambda\Gamma^{(0)}.$$

But we can prove more:

$$\Gamma(L) = \{\chi^\infty P; \theta_1\Gamma^{(0)}, \cdots, \theta_\lambda\Gamma^{(0)}\}.$$

In order to see this we only have to prove the existence of elements in L whose values are equal to $1/p^j$ for each $p \neq \chi$ and $j = 1, 2, \cdots$. We apply for this purpose an argument which is used in the theory of Puiseux developments of algebraic function fields of two variables. Namely, consider the subfield $\boldsymbol{k}(x_0, y)$ of two variables. It is contained in the perfect field M of all formal power series $\sum_{i=0}^{\infty} \boldsymbol{b}_i x_0^{s_i}$ where $\boldsymbol{b}_i \in \boldsymbol{k}$ and $\{s_i\} \rightarrow \infty$ are arbitrary sequences of rational numbers. The perfectness of this field can readily be proved. The element $y = \Sigma \boldsymbol{a}_i x_0^{r_i}$ which is transcendental over $\boldsymbol{k}(x_0)$ is contained in the field M. We have to show that suitable rational functions of x_0 and y have the values $1/p^j$. Let then $\boldsymbol{a}_i x^{g_i/p^j}$ be *the* term in $y = \Sigma \boldsymbol{a}_i x_0^{r_i}$ whose denominator is exactly equal to p^j, according to our assumption on the sequence $\{r_i\}$ such a term is uniquely determined. We next suppose, in order to apply induction with regard to i, that we already proved the existence of rational functions of x_0, y whose values are equal to $1/p^l$ where p and l belong to the exponents r_ν, $\nu < j$. We write the series $y = \Sigma \boldsymbol{a}_i x_0^{r_i}$ in the form

$$y = \sum_{\mu=0}^{i-2} \boldsymbol{a}_\mu x_0^{r_\mu} + \boldsymbol{a}_{i-1} x_0^{r_{i-1}} + \boldsymbol{a}_i x_0^{r_i} + \sum_{\mu=i+1}^{\infty} \boldsymbol{a}_\mu x_0^{r_\mu}$$
$$= S' + \boldsymbol{a}_{i-1} x_0^{r_{i-1}} + \boldsymbol{a}_i x_0^{r_i} + S''.$$

Next we consider the element

$$\bar{y} = S' + \boldsymbol{a}_{i-1} x_0^{r_{i-1}}$$

and its conjugates

$$\bar{y}_\rho = S'^{\sigma_\rho} + \boldsymbol{a}_{i-1} (x_0^{r_{i-1}})^{\sigma_\rho}$$

where σ_ρ denote the $q_{i-1}^{l_{i-1}} = \tau$ automorphisms of the field M given by $x_0^{1/\tau} \rightarrow \zeta_\tau x_0^{1/\tau}$, ζ_τ denoting a fixed τ-th root of unity. $(r_{i-1} = r'_{i-1}/\tau)$ form the product

$$\prod_{\rho=1}^{\tau} (y - \bar{y}_\rho).$$

As in the classical theory of Puiseux developments it is seen that $\Pi(y - \bar{y}_\rho)$ is an element of $\boldsymbol{k}(x_0, y)$. The first term of the product $\Pi(y - \bar{y}_\rho)$ has then a first term $x_0^{s/m}$ where the denominator m is exactly divisible by p^j. There still may occur powers of other primes in m, however, they belong to the rational numbers r_μ $(\mu = 0, \cdots, i-2)$. We write then s/m as a sum of elementary fractions

$$\sum_{\mu=0}^{i-2} s_\mu p_\mu^{-l_\mu} + s' p^{-l}$$

where p_μ are the denominators of r_μ. According to induction, there exist elements z_μ in $\boldsymbol{k}(x_0, y)$ such that

$$v(z_\mu) = 1/p_\mu{}^{l_\mu}.$$

The element

$$\prod_{\mu=0}^{i-2} z_\mu{}^{-s_\mu} \prod (y - \bar{y}_\rho) = z$$

has the value

$$s'/p^l, \text{ or } z = \boldsymbol{c}x_0{}^{s'/p^l} + S'''$$

where $\boldsymbol{c} \neq 0$ and S''' consists of powers of x_0 whose exponents are greater than s'/p^l and whose denominators satisfy the same conditions as those of S''. Let s'' be a solution of the congruence

$$s's'' \equiv 1 \ (\mathrm{mod}\ p^l), \text{ i.e. } s's'' = 1 + gp^l, \ g \text{ an integer.}$$

The element $z^{s''}$ has the form

$$\boldsymbol{c}^{s''}x_0{}^{1/p^l} + g + S^{(4)},$$

consequently the element $z^{s''}x_0{}^{-g}$ has the value $1/p^l$. Thus we see by repeating this process that any power of a prime $p \neq \chi$ can occur in the denominators of the values belonging to suitable *rational* functions of x_0 and y. Therefore,

$$\Gamma(\boldsymbol{k}(x_0, y)) = \{\mathrm{P}\chi^\infty\}$$

and hence

$$\Gamma(\boldsymbol{k}(x_0, y; x_1, \cdots, x_\lambda)) = \{\mathrm{P}\chi^\infty, \theta_1\Gamma^{(0)}, \cdots, \theta_\lambda\Gamma^{(0)}\}.$$

Again as before we have to distinguish two cases according to the value of the characteristic χ. If $\chi = 0$ then we find that already

$$\Gamma(\boldsymbol{k}(x_0, y; x_1, \cdots, x_\lambda)) = \{\mathrm{P}, \theta_1\Gamma^{(0)}, \cdots, \theta_\lambda\Gamma^{(0)}\}.$$

Consequently, the perfect closure k of $\boldsymbol{k}(x_0, y; x_1, \cdots, x_\lambda)$ has the desired properties. We remark that the perfect closure contains also all formal power series $\Sigma \boldsymbol{b}_i x_0{}^{R_i}$ where $\boldsymbol{b}_i \varepsilon \boldsymbol{k}$ and $R_i \to \infty$ denotes any sequence of rational numbers whose denominators are relatively prime to χ. If $\chi \neq 0$ then we form the infinite algebraic extension N of $\boldsymbol{k}(x_0, y; x_1, \cdots, x_\lambda)$ which contains all solutions of the equations

$$y_\nu{}^\chi \dot{-} y_\nu = a_\nu, \qquad (\nu = 1, 2, \cdots).$$

The valuation $\mathfrak{p}$ of $\boldsymbol{k}(x_0, y; x_1, \cdots, x_\lambda)$ has then an extension $\mathfrak{P}$ in N whose value group is equal to $\{\mathrm{P}, \theta_1\Gamma^{(0)}, \cdots, \theta_\lambda\Gamma^{(0)}\}$ and whose residue class field is equal to $\boldsymbol{k}$.

The last construction can also be applied if the field of residue classes $\boldsymbol{k}$ is not algebraically closed. We start again with a series $y = \Sigma \boldsymbol{a}_i x_0^{r_i}$, then we extend $\boldsymbol{k}(x_0, y)$ to its unramified algebraic closure $\bar{\boldsymbol{k}}(x_0, y)$ and apply the previous considerations to $\boldsymbol{k}(x_0, y)$. We find that $\Gamma(\bar{\boldsymbol{k}}(x_0, y)) = \{\chi^\infty \mathrm{P}\}$. Since $\bar{\boldsymbol{k}}(x_0, y)$ is an unramified extension of $\boldsymbol{k}(x_0, y)$ with respect to the valuation as given by $v(x_0) = 1$, $v(y) = r_0$ we have

$$\Gamma(\boldsymbol{k}(x_0, y)) = \Gamma(\bar{\boldsymbol{k}}(x_0, y)), \text{ i. e.}$$

there exist rational functions of x_0, y whose values are equal to $1/p^l$ where $(p, \chi) = 1$.

In order to construct perfect fields of characteristic 0 with prescribed value group where fields of residue classes have characteristic χ, we start with a χ-adic number field k and adjoin transcendental quantities $x_1, \cdots, x_\lambda$ to it. It is evident how the construction can be completed. Also it is fairly obvious how to construct perfect fields k whose value groups $\Gamma(k)$ have the type

$$\{G_0^{-1}\Gamma^{(0)}, G_1^{-1}\Gamma^{(0)}\theta_1, \cdots, G_\lambda^{-1}\Gamma^{(0)}\theta_\lambda\}$$

where the G_i are Steinitz-numbers describing the possible denominators of the respective components.[25] Incidentally, if $G_0 G_1 \cdots G_\lambda = \prod_p p^\infty$ and

$$(G_i, G_j) = 1 \text{ when } i \neq j$$

the usual local class field theory holds over k if the field of residue classes admits for each integer n exactly one cyclic extension of degree n, as in the preceding sections we have to restrict our considerations to fields and algebras whose degrees are relatively prime to the characteristic.

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[25] E. Steinitz, "Algebraische Theorie der Körper," *Crelle*, vol. 137 (1910).

THEORY OF MULTIGROUPS.*

By Melvin Dresher and Oystein Ore.

Various problems in non-commutative algebra lead naturally to the introduction of algebraic systems in which the operations are not one-valued. Let us only mention co-set expansions in groups with respect to non-normal subgroups or residue systems of one-sided ideals in rings. In the following we shall discuss the general theory of multigroups, i. e., algebraic systems satisfying all the axioms of a group except that the multiplication is multivalued.

The history of the theory of multigroups is very short. Multigroups (or hypergroups) were first defined by *Marty* (1934), who has studied their properties and applications in several communications.[1] Another contribution to this theory was made last year by H. S. *Wall*.[2] In addition one should mention two short notes by *Kuntzmann* [3] and some considerations by *Ore* [4] on co-set expansions in groups and their properties as multigroups.[4a]

In the following we have adopted the most general definition of a multigroup. We assume only the existence of solutions of linear relations and not the existence of two-sided units and two-sided inverses as in most previous investigations. We also find that the theory appears more elegant in this general form.

The first chapter contains a discussion of the definitions and representations of multigroups, together with examples. One also finds a theory of

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* Received February 28, 1938.

[1] F. Marty, (1) "Sur une généralisation de la notion de groupe," *Huitième congrès des mathématiciens scandinaves*, Stockholm, 1934, pp. 45-49; (2) "Rôle de la notion d'hypergroupe dans l'étude des groupes non abéliens," *Comptes Rendus*, vol. 201 (Paris, 1935), pp. 636-638; (3) "Sur les groupes et hypergroupes attachés à une fraction rationelle," *Annales de l'école normale*, 3 sér., vol. 53 (1936), pp. 83-123.

[2] H. S. Wall, "Hypergroups," *American Journal of Mathematics*, vol. 59 (1937), pp. 77-98.

[3] J. Kuntzmann, (1) "Opérations multiformes. Hypergroupes," *Comptes Rendus*, vol. 204 (Paris, 1937), pp. 1787-1788; (2) "Homomorphie entre systèmes multiformes," *Comptes rendus*, vol. 205 (Paris, 1937), pp. 208-210.

[4] Oystein Ore, "Structures and group theory, I," *Duke Mathematical Journal*, vol. 3 (1937), pp. 149-174.

[4a] Added in proofs: Further contributions have appeared since the completion of this paper: L. W. Griffiths, "On hypergroups, multigroups and product systems," *American Journal of Mathematics*, vol. 60 (1938), pp. 345-354; M. Krasner, "Sur la primitivité des corps $\mathfrak{P}$-adiques," *Mathematica*, vol. 13 (1937), pp. 72-191.

scalars, i. e., elements for which the multiplication is unique. The concept of a scalar was first introduced by Wall. Here we extend his results in various ways.

In the second chapter we consider the properties of sub-multigroups. Usually they do not possess properties corresponding to those of subgroups in groups. The new concepts of *closure* and *reversibility* are introduced, and it is shown that for reversible sub-multigroups there exists co-set expansions and quotient multigroups. These investigations are connected with those of Marty and Kuntzmann on the homomorphisms of multigroups.

The last chapter deals with normal sub-multigroups and the decomposition theorems for multigroups. One can define two important types of normality in a multigroup. The weaker form for normality is sufficient to obtain analogues of the first and second theorem of isomorphism and the theorem of Jordan-Hölder. The stronger form of normality can be shown to be equivalent to the condition that the quotient multigroup shall be an ordinary group. Strongly normal sub-multigroups can only exist in multigroups which are homomorphic to ordinary groups. These investigations finally lead to a very interesting type of multigroups which we call *ultragroups.* These ultragroups are characterized by the property that they have composition series in which every quotient multigroup is an ordinary group.

CHAPTER I. General multigroups.

1. Definitions. A *multigroup* $\mathfrak{M}$ is an algebraic system with one operation called *multiplication.* This multiplication usually satisfies the ordinary group axioms except that *the product is not unique.* We enumerate the axioms as follows:

I. *The product.* If a and b are two elements of $\mathfrak{M}$ then the product $a \cdot b$ (simpler ab) is a sub-set of $\mathfrak{M}$

$$a \cdot b = (c_1, c_2, \cdots). \tag{1}$$

Let us observe explicitly that we make no assumptions on the number of elements in the product; it may be arbitrary and vary from product to product. We also extend the definition of a product to arbitrary subsets. If

$$A = (a_1, a_2, \cdots)$$
$$B = (b_1, b_2, \cdots)$$

then

$$A \cdot B = (\cdots, a_i b_j, \cdots) = \Sigma a_i b_j$$

is the set consisting of all elements of $\mathfrak{M}$ contained in some product $a_i b_j$. We shall write $A \supset a$, when a is an element of a set A.

II. *The associative law.* For any three elements of $\mathfrak{M}$ we have

$$(ab)c = a(bc) = abc.$$

These products have a meaning according to the definition of products of sub-sets.

III. *The quotient axiom.* To any two elements a and b there shall exist other elements x and y such that

$$(2) \qquad ax \supset b, \qquad ya \supset b.$$

Any system whose elements satisfy these axioms we shall call a multigroup. If only the first relation (2) has a solution we shall call $\mathfrak{M}$ a *right multigroup* and if only the second has a solution we have a *left multigroup.*

An element e such that

$$ea \supset a$$

for all a in $\mathfrak{M}$ shall be called a *left unit,* and a *right unit* is defined analogously. A *unit* is an element e such that

$$ea \supset a, \qquad ae \supset a$$

for all a. If

$$e \cdot a = a$$

for all a, then e shall be called a *left scalar unit.* The *right scalar units* are defined in the same manner. If

$$ae = ea = a$$

for all a then e is a *scalar unit* or *absolute unit.*

THEOREM 1. *Let the multigroup $\mathfrak{M}$ contain a left (right) scalar unit e. If there exists any right (left) unit in $\mathfrak{M}$ then it is unique and equal to e and e is the only left (right) scalar unit of $\mathfrak{M}$. If $\mathfrak{M}$ contains an absolute unit then there are no other units.*

Proof. Let e_r be a right unit. Then

$$e \cdot e_r = e_r \supset e$$

and hence $e_r = e$. Any other left scalar unit would also have to be equal to e.

We shall say that a^{-1} is a *left inverse* of a when

$$a^{-1} \cdot a \supset e$$

where e is some unit element. The left inverse depends on the unit element e and even for a fixed e an element will usually have several inverses. We shall usually assume that by left inverses the corresponding e is a left unit. An inverse must always exist according to the quotient axiom. One defines *right inverses* in a similar manner and we shall associate a right inverse with a right unit element. A *two-sided inverse* a^{-1} has the properties

$$a^{-1} \cdot a \supset e, \qquad a \cdot a^{-1} \supset e$$

where e is some (two-sided) unit element. Such inverses are not postulated in our multigroups.

One finds easily that if every product in a multigroup contains but one element the multigroup is a group.

A multigroup can always be imbedded in a *groupoid*, i. e. a system with a unique operation. Let us define a *complex* in $\mathfrak{M}$ to be a subset obtained by forming a product of a finite number of elements. The product of two such complexes is again a complex and the multiplication satisfies the associative law. One can also imbed a multigroup in a ring by writing (1) in the form

$$a \cdot b = c_1 + c_2 + \cdots.$$

By introducing rational integral coefficients to indicate the multiplicity of the occurring terms one can define the sum (and difference) of two complexes.

We shall also say that two multigroups $\mathfrak{M}$ and $\mathfrak{M}'$ are *isomorphic* if there exists a one-to-one correspondence between their elements such that if $a \rightleftarrows a'$ then $ab \supset c$ implies $a'b' \supset c'$.

The definition of a multigroup which we have given here and which we shall adopt in the following, corresponds to the definition of hypergroups by *Marty*. Almost all the investigations on multigroups by *Marty* and by *Wall* and *Kuntzmann* presuppose a stronger definition of a multigroup. Instead of the quotient axiom one assumes:

IV. There exists (two-sided) unit elements.

V. Every element has two-sided inverses.

Note: It might be mentioned that one can have multigroups in which every element is a unit. Such multigroups might be called *unit multigroups*.

Any multigroup satisfying these conditions may be called a *regular multigroup*. It is obvious that the conditions IV and V imply the existence of solutions of the relations (2). From

$$a^{-1} \cdot a \supset e, \qquad a \cdot a^{-1} \supset e$$

where e is a unit follows namely

$$b \cdot a^{-1} \cdot a \supset b, \qquad a \cdot a^{-1} \cdot b \supset b$$

and hence $b \cdot a^{-1}$ contains an element y and $a^{-1} \cdot b$ an element x such that (2) holds.

2. Representations of multigroups. The multiplication in a finite multigroup may be represented by a Cayley square much in the same way as for a finite group. The only difference is that the various places in the table will contain several instead of a single element. We shall only illustrate this by an example of a multigroup of order 3.

	e	a	b
e	e	a, b	a, b
a	a	e, b	e, b
b	b	e, a	e, a

One finds that the associative law is satisfied. It is worth noting that $xa = xb$ for all x.

If $\mathfrak{M}$ is a multigroup of order n one can represent $\mathfrak{M}$ isomorphically by means of generalized permutations in which each index corresponds to a set of numbers. Let $a_1, \cdots, a_n$ be the elements of $\mathfrak{M}$. To an element b of $\mathfrak{M}$ we let correspond the symbols

$$P_b \to \begin{pmatrix} a_1, & \cdots, a_n \\ a_1 b, & \cdots, a_n b \end{pmatrix} \to \begin{pmatrix} 1, & 2, & \cdots, n \\ B_2, & B_2, & \cdots, B_n \end{pmatrix}$$

where B_i contains the indices of those elements of $\mathfrak{M}$ which are contained in $a_i b$. Such a substitution P_b is a unit if and only if B_i contains i for every i. If in the same manner

$$P_c \to \begin{pmatrix} a_1, & \cdots, a_n \\ a_1 c, & \cdots, a_n c \end{pmatrix} \to \begin{pmatrix} 1, & 2, & \cdots, n \\ C_1, & C_2, & \cdots, C_n \end{pmatrix}$$

then we define the product substitution

$$P_{bc} \to \begin{pmatrix} a_1, & \cdots, a_n \\ a_1 bc, & \cdots, a_n bc \end{pmatrix} \to \begin{pmatrix} 1, 2, \cdots, n \\ P_b : C_1, \; P_b : C_2, \cdots, P_b : C_n \end{pmatrix}.$$

By this definition one obtains an isomorphic representation of the multigroup.

One can also define a *matrix representation* of $\mathfrak{M}$ in a manner which corresponds to the ordinary regular representation of a group. One lets a correspond to a matrix

$$a \to A_a = (\delta_{i,j}^{(a)})$$

where

$$\delta_{i,j}^{(a)} = 1 \text{ when } aa_i \supset a_j$$
$$\delta_{i,j}^{(a)} = 0 \text{ otherwise.}$$

One finds that if

$$a \cdot b = (c_1, c_2, \cdots)$$

then one has for the corresponding matrices

$$A_a \cdot A_b = A_{c_1} + A_{c_2} + \cdots.$$

We shall not discuss further the properties of this representation.[5]

3. Scalars. We shall say that an element α is a *left scalar* when αx is a single element for all x. A *right scalar* is defined similarly. When αx and $x\alpha$ are single elements for all x we shall say that α is a *scalar*. Obviously the product of any two scalars of same type is again a scalar of the same type.

For the moment we shall consider only left scalars. We shall say that a product ab is a left scalar when abx is a single element for all x.

THEOREM 2. *When a product ab is a left scalar then a is a left scalar and $a \cdot b = x$ is a left scalar element.*

Proof. Since every element in $\mathfrak{M}$ is contained in some product bx we have that ay is single element for all y. Hence a is a left scalar and ab is a single element.

We shall say that the *cancellation law* holds for a left scalar α if any relation

$$\alpha x = \alpha y$$

implies $x = y$. In a finite multigroup the cancellation law is always satisfied because any element is representable in the form αx so that the two sets $\{x\}$ and $\{\alpha x\}$ contain the same number of elements.

We shall now prove:

THEOREM 3. *Let $\mathfrak{M}$ be a multigroup in which the cancellation law holds for left scalars. The existence of a left scalar then implies the existence of a left scalar unit.*

Proof. Let α be a left scalar and let us determine an e_α such that

$$\alpha \cdot e_\alpha = \alpha. \qquad (3)$$

A III

Then one finds for an arbitrary x

[5] Compare Wall, *loc. cit.* (part I).

$$\alpha \cdot e_a \cdot x = \alpha x.$$

If now y is some element in $e_a x$ then $\alpha y = \alpha x$ or $y = x$. This gives

(4) $$e_a \cdot x = x$$

for all x.

Under the assumptions of Theorem 3 one can also show:

THEOREM 4. *To each left scalar α there exists a unique left scalar inverse α^{-1} such that*

(5) $$\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = e_a.$$

Proof. The cancellation law gives the existence of a unique α^{-1} such that

$$\alpha \cdot \alpha^{-1} = e_a.$$

From this relation follows

$$\alpha (\alpha^{-1}\alpha) = e_a \cdot \alpha = \alpha = \alpha \cdot e_a$$

and hence we have

$$\alpha \cdot y = \alpha \cdot e_a$$

or $y = e_a$ for every y in $\alpha^{-1} \cdot \alpha$. Theorem 2 implies that α^{-1} is a left scalar.

When the cancellation law holds for left scalars one can strengthen Theorem 2 to:

THEOREM 5. *When ab is a left scalar then both a and b are left scalars.*

Proof. If bx contains c_1 and c_2 then $ac_1 = ac_2$ or $c_1 = c_2$. Hence bx is a single element.

THEOREM 6. *Let $\mathfrak{M}$ be a multigroup with a right unit r, in which the cancellation law holds for left scalars. If $\mathfrak{M}$ contains a left scalar then the set of all left scalars is a group with r as unit element.*

Proof. It follows from Theorem 1 that r is the only right unit and the only left scalar unit of $\mathfrak{M}$. The group property of the scalars follows from the two preceding theorems.

Let us now consider multigroups with (two-sided) scalars.[6]

THEOREM 6. *If a multigroup contains a scalar element then it contains an absolute unit element e and the set of all scalars is a group with e as unit element.*

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[6] These are the only scalars considered by Wall.

Proof. Let α be the given scalar. We determine e_a such that

$$\alpha \cdot e_a = \alpha.$$

Then one has for any x

$$x\alpha \cdot e_a = x_\alpha.$$

Since every element of $\mathfrak{M}$ is representable in the form $x\alpha$ this means that e_a is a left scalar unit. In the same manner one establishes the existence of a right scalar unit and hence an absolute unit according to Theorem 1.

To prove that the scalars form a group let us determine α^{-1} such that

$$\alpha \cdot \alpha^{-1} = e.$$

Then

$$\alpha \cdot \alpha^{-1} \cdot \alpha = \alpha$$

and as before this implies

$$y \cdot \alpha^{-1}\alpha = y$$

for all y. Hence we have

$$\alpha^{-1} \cdot \alpha = \alpha \cdot \alpha^{-1} = e.$$

The inverse is a scalar according to Theorem 2. It is unique because any relation $\alpha x = \alpha y$ implies $x = y$ as one sees by multiplication with α^{-1}.

The group of scalars has been called the *nucleus* of $\mathfrak{M}$ by *Wall.*

THEOREM 7. *Let $\mathfrak{M}$ be a finite multigroup. If $\mathfrak{M}$ contains both a right and a left scalar then all scalars are two-sided.*

Proof. Since $\mathfrak{M}$ is finite the cancellation law holds and $\mathfrak{M}$ contains right and left scalar units according to Theorem 3. This implies the existence of an absolute unit element by Theorem 1. Furthermore we have for any left scalar α

$$\alpha^{-1} \cdot \alpha = \alpha \cdot \alpha^{-1} = e$$

and Theorem 2 shows that α is a scalar.

It is also easily seen:

THEOREM 8. *In any regular multigroup with an absolute unit element every left or right scalar is a scalar.*

4. Examples. The decompositions of ordinary groups furnish us with some of the most important examples of multigroups. Let us consider the quotient systems in groups as our first application.[7]

Let $\mathfrak{G}$ be a group and $\mathfrak{H}$ any subgroup. The co-sets $a\mathfrak{H}$ (or $\mathfrak{H}a$) form a

[7] Compare O. Ore, *loc. cit.*, chap. I.

set which we shall call the *quotient system* $\mathfrak{G}/\mathfrak{H}$ of $\mathfrak{G}$ with respect to $\mathfrak{H}$. This quotient system is a multigroup when one defines the product to be

$$a\mathfrak{H} \cdot b\mathfrak{H} = (c_1\mathfrak{H}, c_2\mathfrak{H}, \cdot \cdot \cdot)$$

where the $c_i\mathfrak{H}$ are those co-sets which contain elements from the left-hand product. The multigroup is an ordinary group if and only if $\mathfrak{H}$ is normal in $\mathfrak{G}$. We shall assume that this is not the case. This is equivalent to saying that the normaliser group $\mathfrak{N}$ of $\mathfrak{H}$ in $\mathfrak{G}$ is a proper subgroup of $\mathfrak{G}$.

Under this assumption we find: *The only unit element in $\mathfrak{G}/\mathfrak{H}$ is $\mathfrak{H}$ and*

$$a\mathfrak{H} \cdot \mathfrak{H} = a\mathfrak{H}, \qquad \mathfrak{H} \cdot a\mathfrak{H} \supset a\mathfrak{H}.$$

If $b \cdot \mathfrak{H}$ is to be a r. h. inverse of $a\mathfrak{H}$ then one must have

$$a\mathfrak{H} \cdot b\mathfrak{H} \supset \mathfrak{H},$$

hence there must exist elements h_0 and h_1 in $\mathfrak{H}$ such that

$$ah_0b = h_1, \qquad b = h_1 \cdot a^{-1} \cdot h_0^{-1}. \quad ?$$

It then follows easily: *Any inverse of $a\mathfrak{H}$ has the form $ha^{-1} \cdot \mathfrak{H}$ where h belongs to $\mathfrak{H}$. These inverses are all both r. h. and l. h. inverses, hence $\mathfrak{G}/\mathfrak{H}$ is regular multigroup.*

To determine the scalars in $\mathfrak{G}/\mathfrak{H}$ let us consider the condition for an equation

(9) $$a\mathfrak{H} \cdot b\mathfrak{H} = c\mathfrak{H}.$$

to hold. One sees that c must have the form

$$c = ab \cdot h$$

and when this value is substituted in (9) one finds

$$b^{-1}\mathfrak{H}b = \mathfrak{H}.$$

Hence a relation (9) holds if and only if $b\mathfrak{H}$ belongs to the normaliser of $\mathfrak{H}$ in $\mathfrak{G}$. This shows: *The only right-hand scalars of $\mathfrak{G}/\mathfrak{H}$ are the co-sets of the normaliser of $\mathfrak{H}$ in $\mathfrak{G}$. There are no left-hand scalars.*

Another important type of multigroups is defined by the co-set expansion of a group $\mathfrak{G}$ with respect to two subgroups $\mathfrak{M}$ and $\mathfrak{N}$

$$\mathfrak{G} = \mathfrak{M}\,\mathfrak{N} + \mathfrak{M}a_2\mathfrak{N} + \cdot \cdot \cdot .$$

These double co-sets form a multigroup when one defines

$$\mathfrak{M}a\mathfrak{N} \cdot \mathfrak{M}b\mathfrak{N} = (\mathfrak{M}c_1\mathfrak{N}, \mathfrak{M}c_2\mathfrak{N}, \cdot \cdot \cdot)$$

where as before the right-hand side contains those co-sets which include elements from the l. h. product. One might denote this multigroup by $\mathfrak{M}\diagdown\mathfrak{G}/\mathfrak{N}$. One can determine its units and inverses and one finds that it is a regular multigroup. The results are somewhat complicated and they shall not be reproduced here. It may be of interest to note that scalars only exist when the two groups are permutable.

In the simplest case where $\mathfrak{M} = \mathfrak{N}$ let us denote the multigroup $\mathfrak{N}\diagdown\mathfrak{G}/\mathfrak{N}$ by $\mathfrak{G}/\!\!/\mathfrak{N}$. This multigroup has the absolute unit $\mathfrak{N}$. The inverse of $\mathfrak{N}a\mathfrak{N}$ is unique and equal to $\mathfrak{N}a^{-1}\mathfrak{N}$. Again the scalars are the co-sets belonging to the normaliser of $\mathfrak{N}$ in $\mathfrak{G}$.

CHAPTER 2. Co-set expansions.

1. Closed sub-multigroups. Let $\mathfrak{M}$ be a multigroup. We shall say that a sub-set $\mathfrak{A}$ is *multiplicatively closed* when it has the property that if a_1 and a_2 belong to $\mathfrak{A}$ then all elements contained in the product a_1a_2 also belong to $\mathfrak{A}$. We shall say that $\mathfrak{A}$ is a *sub-multigroup* of $\mathfrak{M}$, if it satisfies the axioms of a multigroup, i. e. if the relations

$$a_1x \supset a_2, \qquad ya_1 \supset a_2$$

always have solutions in $\mathfrak{A}$.

When $\mathfrak{A}$ and $\mathfrak{B}$ are two sub-multigroups then the *cross-cut* $(\mathfrak{A}, \mathfrak{B})$ of their common elements is multiplicatively closed. Usually $(\mathfrak{A}, \mathfrak{B})$ is not a sub-multigroup, because if d_1 and d_2 are two of its elements the relations

$$d_1x \supset d_2, \qquad yd_1 \supset d_2$$

need not have solutions in $(\mathfrak{A}, \mathfrak{B})$.

This leads us to the consideration of more special types of sub-multigroups. We shall say that a sub-multigroup $\mathfrak{A}$ is *left closed* with respect to $\mathfrak{M}$ if for any two elements a_1 and a_2 in $\mathfrak{A}$ all the solutions y of the relation

$$ya_1 \supset a_2 \tag{1}$$

lie in $\mathfrak{A}$. Similarly we say that $\mathfrak{A}$ is *right closed* when all the solutions of

$$a_1x \supset a_2 \tag{2}$$

lie in $\mathfrak{A}$. Finally $\mathfrak{A}$ is *closed* when all the solutions of (1) and (2) lie in $\mathfrak{A}$.

One sees immediately:

THEOREM 1. *The cross-cut* $(\mathfrak{A}, \mathfrak{B})$ *of two (left) closed sub-multigroups is void or a (left) closed sub-multigroup.*

The definition shows that a (left) closed sub-multigroup must contain all (left) units of $\mathfrak{M}$. Hence if $\mathfrak{M}$ contains left units the cross-cut of two left closed sub-multigroups is never void. In the following we shall usually assume the existence of such a unit element. Let us also observe that any (left) closed sub-multigroup contains all its (left) inverses.

The *union* $[\mathfrak{A}, \mathfrak{B}]$ of two sub-multigroups (or multiplicatively closed sub-sets) consists of all those elements which are contained in some product made up of factors from $\mathfrak{A}$ and $\mathfrak{B}$. The union is multiplicatively closed, but it is usually not a sub-multigroup even if $\mathfrak{A}$ and $\mathfrak{B}$ are closed. These remarks show that if the void set is included in the multiplicatively closed sets, one has:

THEOREM 2. *The set of all multiplicatively closed sub-systems form a structure.*

A similar theorem does usually not hold for the sub-multigroups.

One finds that in the multigroups defined by ordinary co-set expansions $\mathfrak{G}/\mathfrak{H}$ every sub-multigroup is closed. The same holds for the double co-set expansions $\mathfrak{G}/\!/\mathfrak{H}$.

2. Reversibility. Let $\mathfrak{M}$ be a multigroup. We shall say that a sub-multigroup $\mathfrak{A}$ of $\mathfrak{M}$ is *left reversible in* $\mathfrak{M}$ if any relation

$$a_1 m_1 \supset m_2$$

$a_1 \in A$
$m_1 \in M$

where a_1 belongs to $\mathfrak{A}$ implies the existence of an a_2 in $\mathfrak{A}$ such that

$$m_1 \subset a_2 m_2.$$

When m_2 is taken as an element in $\mathfrak{A}$ one finds that m_1 is an element of $\mathfrak{A}$, hence:

THEOREM 3. *Left reversibility implies right closure.*

Similarly we say that $\mathfrak{A}$ is *right reversible* in $\mathfrak{M}$, when any relation

$$m_1 a_1 \supset m_2$$

implies the existence of an a_2 in $\mathfrak{A}$ such that

$$m_1 \subset m_2 a_2.$$

Finally $\mathfrak{A}$ is *reversible* in $\mathfrak{M}$ when it is both left and right reversible. According to Theorem 3 a reversible sub-multigroup is closed.

We can now prove:

THEOREM 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be closed and $\mathfrak{A}$ (left) reversible in $\mathfrak{M}$. Then $(\mathfrak{A}, \mathfrak{B})$ is closed and (left) reversible in $\mathfrak{B}$.*

Proof. According to Theorem 1, $(\mathfrak{A}, \mathfrak{B})$ is closed. Furthermore since $\mathfrak{A}$ is left reversible any relation

$$d_1 b_1 \supset b_2$$

where d_1 belongs to $(\mathfrak{A}, \mathfrak{B})$, implies that

$$b_1 \subseteq a_2 \cdot b_2$$

and since $\mathfrak{B}$ is closed we have $a_2 = d_2$.

We are now able to prove:

THEOREM 5. *The union of two reversible multigroups is a reversible multigroup.*

Proof. If c_1 and c_2 are arbitrary elements in $[\mathfrak{A}, \mathfrak{B}]$ then we shall first have to prove that any relations

$$xc_1 \supset c_2, \qquad c_1 y \supset c_2 \tag{3}$$

imply that x and y be in $[\mathfrak{A}, \mathfrak{B}]$. It is sufficient to consider the first of these relations. Each element of the union is contained in some product of factors from $\mathfrak{A}$ and $\mathfrak{B}$. Let us suppose that

$$a_n b_n \cdots a_1 b_1 \supset c_1 \tag{4}$$

is the product of shortest length containing c_1. When c_1 in (3) is contained in $\mathfrak{A}$ or $\mathfrak{B}$ our assertion is true by the reversibility of $\mathfrak{A}$ and $\mathfrak{B}$. Hence we may prove it in general by induction, assuming there exists solutions for all c_1 contained in products of shorter length. From (3) and (4) one obtains

$$xa_n b_n \cdots a_1 b_1 \supset c_2$$

and hence there exists an element d such that

$$x \cdot d \cdot b_1 \supset c_2, \qquad d \subset a_n b_n \cdots a_1.$$

This in turn implies the existence of an x_1 such that

$$x_1 \cdot b_1 \supset c_2, \qquad x_1 \subset xd.$$

The first of these relations show that x_1 belongs to $[\mathfrak{A}, \mathfrak{B}]$ and since d is contained in a product of shorter length than c_1 it follows from the second

that x also belongs to $[\mathfrak{A}, \mathfrak{B}]$. The reversibility of $[\mathfrak{A}, \mathfrak{B}]$ follows in a similar manner.

Now let us define further: Let $\mathfrak{M}$ be a multigroup with (left, right) units. We shall say that $\mathfrak{M}$ is (left, right) *reversible in itself* when any relation

$$m_1 \cdot m_2 \supset m_3$$

implies the existence of a left inverse m_1^{-1} and a right inverse m_2^{-1} such that

$$m_2 \subset m_1^{-1} \cdot m_3, \qquad m_1 \subset m_3 \cdot m_2^{-1}.$$

From this definition follows:

THEOREM 6. *Let $\mathfrak{M}$ be (left) reversible in itself. Then every closed sub-multigroup is left reversible in $\mathfrak{M}$.*

When this theorem is combined with Theorems 4 and 5 one obtains:

THEOREM 7. *Let $\mathfrak{M}$ be a multigroup which is reversible in itself. Then the set of all closed sub-multigroups of $\mathfrak{M}$ form a structure.*

Let us observe that in ordinary groups the co-set expansions $\mathfrak{G}/\mathfrak{H}$ and $\mathfrak{G}/\!/\mathfrak{H}$ define multigroups which are reversible in themselves.

3. Co-set expansions. Let $\mathfrak{A}$ denote some sub-multigroup or even only a multiplicatively closed sub-set of $\mathfrak{M}$. By $\mathfrak{A}m$ we shall denote the set of all elements contained in some product $a \cdot m$ where a belongs to $\mathfrak{A}$. We shall call $\mathfrak{A}m$ a *left co-set* of $\mathfrak{M}$ with respect to $\mathfrak{A}$.

From now on we shall suppose that $\mathfrak{A}$ is left reversible in $\mathfrak{M}$. Since any relation

$$a_1 m_1 \supset m_2$$

implies

$$m_1 \subset a_2 \cdot m_2$$

or

$$\mathfrak{A}m_1 = \mathfrak{A}m_2$$

we can state:

THEOREM 8. *Any element in a co-set $\mathfrak{A}m$ generates the same co-set and a co-set contains its generating element.*

This gives us the main theorem:

THEOREM 9. *Let $\mathfrak{A}$ be a left reversible multiplicatively closed sub-system of a multigroup $\mathfrak{M}$. Then $\mathfrak{M}$ has a unique co-set expansion*

(5) $$\mathfrak{M} = \mathfrak{A}m_1 + \mathfrak{A}m_2 + \cdots$$

where the co-sets $\mathfrak{A}m$ are disjoint sets.

Proof. Every element in $\mathfrak{M}$ belongs to some co-set and two co-sets with a common element are identical according to Theorem 9.

Two co-sets will usually not contain the same number of elements. Let us also observe that the left reversibility of $\mathfrak{A}$ implies that $\mathfrak{A}$ is a right multigroup which is right closed.

The existence of double co-set expansions may be derived under similar conditions. We prove:

THEOREM 10. *Let $\mathfrak{A}$ be a left reversible and $\mathfrak{B}$ a right reversible multiplicatively closed sub-set of $\mathfrak{M}$. Then $\mathfrak{M}$ has a double co-set expansion*

(6) $$\mathfrak{M} = \mathfrak{A}m_1\mathfrak{B} + \mathfrak{A}m_2\mathfrak{B} + \cdots$$

where the co-sets $\mathfrak{A}m\mathfrak{B}$ are disjoint sets.

Proof. As before it suffices to prove that any element in a double co-set generates the same co-set. Now if

$$amb \supset m'$$

then there exists an $\bar{m}$ such that

$$\bar{m} \subset a \cdot m, \qquad m \subset a_1 \cdot \bar{m}$$

and such that

$$\bar{m}b \supset m', \qquad \bar{m} \subset m'b_1.$$

This gives

$$a_1m'b_1 \supset a_1\bar{m} \supset m$$

and our assertion is proved.

As a special case we have:

THEOREM 11. *Let $\mathfrak{A}$ be a sub-multigroup reversible in $\mathfrak{M}$. Then there exists a unique double co-set expansion*

(7) $$\mathfrak{M} = \mathfrak{A}m_1\mathfrak{A} + \mathfrak{A}m_2\mathfrak{A} + \cdots.$$

4. Quotient multigroups.[8] Let us assume that $\mathfrak{A}$ is a left reversible sub-multigroup of $\mathfrak{M}$. Then $\mathfrak{A}$ is seen to be one of the co-sets (5). These co-sets $\mathfrak{A}m$ may be considered to be the elements of a new multigroup which we shall call the *quotient multigroup* of $\mathfrak{M}$ with respect to $\mathfrak{A}$ and denote by $\mathfrak{M}/\mathfrak{A}$. We define the product of two co-sets to be

[8] Compare Marty (3).

$$\mathfrak{A}m_1 \cdot \mathfrak{A}m_2 = (\mathfrak{A}n_1, \mathfrak{A}n_2, \cdot \cdot \cdot)$$

where the co-sets on the right are those which contain elements from the left-hand complexes. To prove that $\mathfrak{M}/\mathfrak{A}$ is a multigroup one must show that the relations

$$\mathfrak{A}m_1 \cdot \mathfrak{A}x \supset \mathfrak{A}m_2, \qquad \mathfrak{A}y \cdot \mathfrak{A}m_1 \supset \mathfrak{A}m_2$$

always have solutions. One sees however that it is sufficient to choose x and y such that

$$m_1 x \supset m_2, \qquad y m_1 \supset m_2.$$

THEOREM 11. *Let $\mathfrak{A}$ be a left reversible sub-multigroup of $\mathfrak{M}$. The left co-set expansion of $\mathfrak{M}$ with respect to $\mathfrak{A}$ defines a quotient multigroup $\mathfrak{M}/\mathfrak{A}$ with the left scalar unit $\mathfrak{A}$.*

One might add that when $\mathfrak{M}$ contains a right unit then it must be contained in $\mathfrak{A}$ since $\mathfrak{A}$ is right closed, and hence $\mathfrak{A}$ is a right unit. It is the only right unit according to Theorem 1, chap. 1. When $\mathfrak{A}$ is closed, then $\mathfrak{A}$ is the only right or left unit.

Now let us consider any multigroup $\mathfrak{B}$ between $\mathfrak{M}$ and $\mathfrak{A}$, i. e.

$$\mathfrak{M} \supset \mathfrak{B} \supset \mathfrak{A}.$$

One can expand $\mathfrak{B}$ in co-sets with respect to $\mathfrak{A}$ and these co-sets obviously form a sub-multigroup $\mathfrak{B}/\mathfrak{A}$ of $\mathfrak{M}/\mathfrak{A}$. One finds without difficulty that if $\mathfrak{B}$ is closed or reversible in $\mathfrak{M}$ in some sense then $\mathfrak{B}/\mathfrak{A}$ has the same property in $\mathfrak{M}/\mathfrak{A}$.

Now let us consider the converse correspondence. Let $\mathfrak{B}^*$ be a sub-multigroup of $\mathfrak{M}/\mathfrak{A}$ containing $\mathfrak{A}$. We can then show that $\mathfrak{B}^*$ when considered as a set of elements in $\mathfrak{M}$ will be a multigroup $\mathfrak{B}$ containing $\mathfrak{A}$. To verify this we shall have to show that the relations

$$x b_1 \supset b_2, \qquad b_1 y \supset b_2 \tag{8}$$

have solutions in $\mathfrak{B}$ when b_1 and b_2 are elements of this set.

To find an x satisfying the first relation (8) let us observe that since $\mathfrak{B}^*$ is a multigroup there exists an x_1 in $\mathfrak{B}$ such that

$$\mathfrak{A}x_1 \cdot \mathfrak{A}b_1 \supset \mathfrak{A}b_2.$$

This implies

$$a_1 x_1 a_2 \cdot b_1 \supset b_2$$

and hence $a_1 x_1 a_2$ contains an element x such that the first relation (8) holds.

To find a solution y of the second relation (8) let us determine z_1 in $\mathfrak{B}$ such that

$$\mathfrak{A}b_1 \cdot \mathfrak{A}z_1 \supset \mathfrak{A}.$$

From this relation follows that one can determine a z_a in $\mathfrak{B}$ such that

$$b_1 \cdot z_a \supset a$$

for an arbitrary a in $\mathfrak{A}$. Since $\mathfrak{A}b_2$ contains b_2 it follows by right multiplication with b_2 that a solution y of (8) in $\mathfrak{B}$ exists.

Let us observe that one can also consider the co-sets of a double co-set expansion (6) or (7) as the elements of a new multigroup $\mathfrak{A}\backslash\mathfrak{M}/\mathfrak{B}$ or $\mathfrak{M}/\!/\mathfrak{A}$. Similar results can be derived for such multigroups.

5. Homomorphisms. We shall say that a multigroup $\mathfrak{M}$ is *homomorphic* to another multigroup $\mathfrak{M}^*$ when there exists a correspondence $m \to m^*$ between the elements of the two systems such that when [9]

$$a \cdot b \supset c$$

then

$$a^* \cdot b^* \supset c^*.$$

Furthermore we suppose that every element of $\mathfrak{M}^*$ is the image of some element of $\mathfrak{M}$.

Obviously the co-set correspondence

$$m \to \mathfrak{A}m = m^*$$

which we have studied, defines a homomorphism of $\mathfrak{M}$ upon $\mathfrak{M}/\mathfrak{A}$. Also the double co-set expansions determine multigroups homomorphic to $\mathfrak{M}$.

$$m \to \mathfrak{A}m\mathfrak{B}, \qquad n \to \mathfrak{A}n\mathfrak{A}.$$

We shall study a few of the properties of such homomorphisms.

THEOREM 12. *All those elements of $\mathfrak{M}$ which correspond to a left scalar unit of $\mathfrak{M}^*$ form a right multigroup which is right closed. All those elements of $\mathfrak{M}$ which correspond to an absolute unit element of $\mathfrak{M}^*$ form a closed submultigroup.*

Proof. First if e^* is an idempotent in $\mathfrak{M}^*$ i. e. an element such that

$$e^{*2} = e^*,$$

[9] Compare Marty (3) and Kuntzmann (2).

then the elements in $\mathfrak{M}$ corresponding to e^* must form a set $\mathfrak{A}$ which is multiplicatively closed. Now let e^* be a left scalar unit, and a_1 and a_2 two elements of $\mathfrak{A}$. From a relation $a_1x \supset a_2$ follows immediately $x^* = e^*$ and x also belongs to $\mathfrak{A}$.

In order to derive further properties of the homomorphism it is necessary to make assumptions on the inverse correspondence from $\mathfrak{M}^*$ to $\mathfrak{M}$. Let us say that there exists a *strong (left) homomorphism* between $\mathfrak{M}$ and $\mathfrak{M}^*$ when the following condition is satisfied.

Let

$$a^*b^* \supset c^*.$$

Then to any b_0 and any c_0 corresponding to b^* and c^* respectively there exists an a corresponding to a^* such that

$$a \cdot b_0 \supset c_0.$$

One sees easily that the correspondence between $\mathfrak{M}$ and a left co-set expansion $\mathfrak{M}/\mathfrak{A}$ is a strong left homomorphism. This need not be true for the two-sided co-set expansion $\mathfrak{M}/\!/\mathfrak{A}$.

We can now prove:

THEOREM 13. *Let $\mathfrak{M}$ be strongly left homomorphic to $\mathfrak{M}^*$ and let $\mathfrak{A}$ be the right closed, right sub-multigroup which consists of the elements corresponding to a left scalar unit e^* of $\mathfrak{M}^*$. Then $\mathfrak{A}$ is left reversible in $\mathfrak{M}$ and $\mathfrak{M}^*$ is isomorphic to the quotient multigroup $\mathfrak{M}/\mathfrak{A}$ defined by the left co-set expansion of $\mathfrak{M}$ with respect to $\mathfrak{A}$.*

Proof. Let us suppose that two elements m_1 and m_2 have the same image. From

$$e^* \cdot m^*_1 = m^*_1 = m^*_2 = e^* \cdot m^*_2$$

follows on account of the strong homomorphism

$$a_1 \cdot m_1 \supset m_2, \qquad m_1 \subset a_2 \cdot m_2$$

where a_1 and a_2 belong to $\mathfrak{A}$. Hence all m with the same image belong to the same left co-set $\mathfrak{A}m$ and two elements of the same co-set obviously have the same image.

If one wishes to characterize also the double co-set expansions one must make a slightly different condition on the homomorphism: If

$$a^* \cdot b^* \cdot c^* \supset d^*$$

then it must be possible to determine to any b_0 and d_0 such an a and such a c that

$$a \cdot b_0 \cdot c \supset d_0.$$

When this condition is satisfied and $\mathfrak{A}$ is the closed sub-multigroup of $\mathfrak{M}$ which corresponds to an absolute unit element of $\mathfrak{M}^*$, then two elements of $\mathfrak{M}$ with the same image belong to the same double co-set with respect to $\mathfrak{A}$.

6. Some special multigroups. Let us conclude this chapter by a short discussion of the properties of those multigroups which are obtained from co-set expansions $\mathfrak{G}/\mathfrak{H}$ in groups. The multigroup defined by the left co-set expansion of a group with respect to a subgroup has a single unit element and is reversible. In addition it satisfies the following condition:

Cancellation law. Let $\mathfrak{M}$ be a multigroup with a single unit element e. We say that $\mathfrak{M}$ satisfies the (*left*) *cancellation law* when any relation

$$ab \supset b$$

implies $a = e$.

We can prove:

THEOREM 8. *A multigroup satisfying the* (*left*) *cancellation law is regular*

Proof. We shall have to prove that every left inverse is a right inverse and conversely. From

$$m^{-1} \cdot m \supset e$$

follows

$$m \cdot m^{-1} \cdot m \supset m$$

or

$$m \cdot m^{-1} \supset e.$$

From this last relation follows conversely

$$m^{-1} \cdot m \cdot m^{-1} \supset m^{-1}$$

or

$$m^{-1} \cdot m \supset e.$$

We can also prove:

THEOREM 9. *A multigroup satisfying the left cancellation law is left reversible.*

Proof. Let us suppose that

(9) $$ab \supset c.$$

We determine x such that $xc \supset b$ and find

$$x \cdot ab \supset xc \supset b.$$

Hence one concludes $x = a^{-1}$ and (9) implies

$$a^{-1} \cdot c \supset b.$$

The double co-set expansions of a group $\mathfrak{G} /\!/ \mathfrak{H}$ have still simpler properties. They belong to the following type of multigroups:

Completely regular multigroups. A multigroup is said to be *completely regular* when it contains a single unit element e and every element has a unique inverse such that

$$m \cdot m^{-1} \supset e, \qquad m^{-1} \cdot m \supset e.$$

For such multigroups one can prove:

THEOREM 10. *In a completely regular, reversible multigroup the right and left co-set expansions contain the same number of co-sets.*

Proof. Let (5) denote the left co-set expansion of $\mathfrak{M}$ with respect to a sub-multigroup $\mathfrak{A}$. We want to show that

$$\mathfrak{M} = m_1^{-1} \cdot \mathfrak{A} + m_2^{-1} \cdot \mathfrak{A} + \cdots \tag{10}$$

is a right co-set expansion. We show first that every element x belongs to some co-set (10). Since x^{-1} belongs to some left co-set, we have

$$am_i \supset x^{-1}$$

which successively implies

$$am_i x \supset e, \qquad m_i x \supset a^{-1}, \qquad x \subset m_i^{-1} \cdot a^{-1}.$$

Secondly no two co-sets (10) are identical since

$$m_i^{-1} \cdot a \supset m_j^{-1}$$

is seen to imply

$$m_j \subset a^{-1} \cdot m_i$$

against assumption.

Let us mention finally that under some very restrictive conditions one can also prove a theorem analogous to the theorem of Lagrange, that the order of a subgroup divides the order of the group.

CHAPTER 3. Decomposition theorems.

1. Normal sub-multigroups and quotient systems. We shall now introduce the concept of a *normal sub-multigroup* or more general, a *normal set* in a multigroup $\mathfrak{M}$. A set $\mathfrak{A}$ is *normal* in $\mathfrak{M}$, if for any m in $\mathfrak{M}$ we have

$$m \cdot \mathfrak{A} = \mathfrak{A} \cdot m$$

i. e. every element contained in a product $m \cdot a_1$ is also contained in some product $a_2 \cdot m$.

For a normal sub-multigroup left or right closure implies closure. Similarly left or right reversibility implies reversibility. Since we usually wish to be able to perform a co-set decomposition of $\mathfrak{M}$ with respect to a normal sub-multigroup $\mathfrak{A}$, we shall in most cases suppose also that $\mathfrak{A}$ is reversible in $\mathfrak{M}$.

One finds easily:

THEOREM 1. *Let $\mathfrak{A}$ be a normal reversible sub-multigroup of $\mathfrak{M}$. The quotient multigroup $\mathfrak{M}/\mathfrak{A}$ is strongly homomorphic to $\mathfrak{M}$ and has the absolute unit $\mathfrak{A}$. Conversely by any strong homomorphism $\mathfrak{M} \rightarrow \mathfrak{M}^*$ those elements of $\mathfrak{M}$ which correspond to the absolute unit element of $\mathfrak{M}^*$ form a normal reversible sub-multigroup $\mathfrak{A}$ of $\mathfrak{M}$ such that $\mathfrak{M}/\mathfrak{A}$ is isomorphic to $\mathfrak{M}^*$.*

Proof. According to the results of § 5, chap. 2 it is only necessary to prove that if e^* is an absolute unit element of $\mathfrak{M}^*$ then those elements $\mathfrak{A}$ of $\mathfrak{M}$ which correspond to e^* form a normal sub-multigroup. Since for any element of $\mathfrak{M}^*$

$$e^* \cdot m^* = m^* = m^* \cdot e^*$$

it follows that all elements with the same image must belong to the co-set $\mathfrak{A}m$ and the co-set $m\mathfrak{A}$.

When $\mathfrak{A}$ is normal and reversible in $\mathfrak{M}$ then the right and left co-set expansions are identical and the quotient system $\mathfrak{M}/\mathfrak{A}$ is uniquely determined. We prove next:

THEOREM 2. *Let $\mathfrak{A}$ be a normal reversible sub-multigroup of $\mathfrak{M}$. Then there exists a one-to-one correspondence between the normal reversible sub-multigroups of $\mathfrak{M}/\mathfrak{A}$ and the normal reversible sub-multigroups of $\mathfrak{M}$ containing $\mathfrak{A}$.*

Proof. It is seen immediately from the definition of $\mathfrak{M}/\mathfrak{A}$ that every normal reversible sub-multigroup $\mathfrak{B}$ between $\mathfrak{M}$ and $\mathfrak{A}$ corresponds to a unique normal reversible sub-multigroup $\mathfrak{B}^*$ of $\mathfrak{M}/\mathfrak{A}$. To prove the converse let $\mathfrak{B}^*$

be a reversible normal sub-multigroup of $\mathfrak{M}/\mathfrak{A}$. Since $\mathfrak{B}^*$ is closed, it contains the unit element $\mathfrak{A}$, hence it corresponds to some multigroup between $\mathfrak{A}$ and $\mathfrak{M}$. From

$$\mathfrak{B} \cdot \mathfrak{A}m = \mathfrak{A}m \cdot \mathfrak{B}$$

we also find

$$\mathfrak{B} \cdot m = m \cdot \mathfrak{B}$$

for any m, hence $\mathfrak{B}$ is normal in $\mathfrak{M}$. To prove finally that any relations

$$b_1 \cdot m_1 \supset m_2, \qquad m_1 \cdot b_2 \supset m_2$$

imply

$$m_1 \subset b_1 \cdot m_2, \qquad m_1 \subset m_2 \cdot b_2$$

it is only necessary to consider the corresponding relations which hold for the co-sets containing these elements.

2. The theorem of isomorphism. We shall now deduce various structural relations for normal sub-multigroups.[10]

THEOREM 3. *Let $\mathfrak{M}$ be a multigroup with units of some kind, and let $\mathfrak{A}$ and $\mathfrak{B}$ be normal reversible sub-multigroups. Then the union $[\mathfrak{A}, \mathfrak{B}]$ is also a normal reversible sub-multigroup and all its elements are contained in some product $a \cdot b$.*

Proof. It follows from the definition of normality that it is only necessary to assume that $\mathfrak{A}$ is normal in order that every element of the union be of the form a or b or contained in a product $a \cdot b$. When $\mathfrak{A}$ contains a left or $\mathfrak{B}$ a right unit of $\mathfrak{M}$, then every element of the union is contained in a product $a \cdot b$. This is always the case when $\mathfrak{M}$ contains some unit element and $\mathfrak{A}$ and $\mathfrak{B}$ are reversible. The normality of the union follows from

$$m \cdot \mathfrak{A} \cdot \mathfrak{B} = \mathfrak{A} \cdot m \cdot \mathfrak{B} = \mathfrak{A} \cdot \mathfrak{B} \cdot m$$

and the reversibility has been proved in Theorem 5, chap. 2.

THEOREM 4. *Let $\mathfrak{M}$ contain some unit element and let $\mathfrak{A}$ and $\mathfrak{B}$ be normal reversible sub-multigroups. Then the cross-cut $(\mathfrak{A}, \mathfrak{B})$ is normal and reversible in $\mathfrak{A}$ and $\mathfrak{B}$.*

Proof. Since $\mathfrak{M}$ contains unit elements the cross-cut $\mathfrak{D} = (\mathfrak{A}, \mathfrak{B})$ is not void. Furthermore $\mathfrak{D}$ is reversible in $\mathfrak{A}$ and in $\mathfrak{B}$ according to Theorem 4,

[10] See also Marty (3).

chap. 2. To prove the normality of $\mathfrak{D}$ in $\mathfrak{B}$ let us observe that since $\mathfrak{D}$ is contained in $\mathfrak{A}$ we have

$$b\mathfrak{D} \subset \mathfrak{A}b$$

and since $\mathfrak{B}$ is closed one finds

$$b\mathfrak{D} \subset \mathfrak{D}b$$

and the equality is obtained by the converse argument.

THEOREM 5. *The Dedekind relation. Let $\mathfrak{M}$ contain some unit element and let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be normal reversible sub-multigroups such that $\mathfrak{C} \supset \mathfrak{A}$. Then*

$$(1) \qquad (\mathfrak{C}, [\mathfrak{A}, \mathfrak{B}]) = [\mathfrak{A}, (\mathfrak{C}, \mathfrak{B})].$$

Proof. The right-hand side of (1) is obviously always contained in the left-hand side. To prove that the left-hand side is contained in the right-hand side we need actually only that $\mathfrak{A}$ is normal and reversible. Any element c contained in the left-hand side then satisfies a relation

$$c \subset a \cdot b$$

and since $\mathfrak{A}$ is reversible it follows that $b = d$ is contained in $\mathfrak{C}$. Hence we have

$$c \subset a \cdot d$$

where d is contained in $(\mathfrak{B}, \mathfrak{C})$.

Finally we prove the *theorem of isomorphism:*

THEOREM 6. *Let $\mathfrak{A}$ be normal and reversible and $\mathfrak{B}$ closed in the union $[\mathfrak{A}, \mathfrak{B}]$. We suppose further that the cross-cut $(\mathfrak{A}, \mathfrak{B})$ is not void. Then $(\mathfrak{A}, \mathfrak{B})$ is normal and reversible in $\mathfrak{B}$ and there exists an isomorphism between the quotient systems*

$$(2) \qquad [\mathfrak{A}, \mathfrak{B}]/\mathfrak{A} \cong \mathfrak{B}/(\mathfrak{A}, \mathfrak{B}).$$

Proof. It follows from Theorem 4, chap. 2 that $(\mathfrak{A}, \mathfrak{B}) = \mathfrak{D}$ is reversible in $\mathfrak{B}$. The normality of $\mathfrak{D}$ in $\mathfrak{B}$ follows as in Theorem 4. Now let

$$(3) \qquad \mathfrak{B} = \mathfrak{D} + b_2\mathfrak{D} + \cdots$$

be the co-set expansion of $\mathfrak{B}$ with respect to $\mathfrak{D}$. We can then show that

$$(4) \qquad [\mathfrak{A}, \mathfrak{B}] = \mathfrak{A} + b_2\mathfrak{A} + \cdots$$

is the co-set expansion of the union with respect to $\mathfrak{A}$. First every element of $[\mathfrak{A}, \mathfrak{B}]$ must belong to one of the co-sets (4) since every b belongs to some co-set (3). Secondly any relation

$$b_i\mathfrak{A} = b_j\mathfrak{A} \qquad i \neq j$$

would imply that

$$b_i \subset b_j \cdot a$$

where a belongs both to $\mathfrak{A}$ and $\mathfrak{B}$, hence to $\mathfrak{D}$ since $\mathfrak{B}$ is closed. This is impossible however since b_i and b_j belong to different co-sets (3). Finally one sees that the correspondence establishes the isomorphism (2).

3. The theorem of Jordan-Hölder. The preceding results are sufficient to prove the analogue of the theorem of *Jordan-Hölder* for multigroups. A chain of sub-multigroups

$$\mathfrak{A} \supset \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_n = \mathfrak{B} \tag{5}$$

shall be called a *composition series* when each $\mathfrak{A}_i$ is normal and reversible in the preceding. We shall call (5) a *maximal composition series,* when no further terms can be intercalated such that the series remains a composition series.

THEOREM 7. *Any two maximal composition series between $\mathfrak{A}$ and $\mathfrak{B}$ have the same length and the quotient multigroups defined by consecutive terms in one series are isomorphic to those defined by the other in some order.*

Proof. The proof follows along the lines of the ordinary proof of the theorem of Jordan-Hölder using induction with regard to the length of the chain. If (5) and

$$\mathfrak{A} \supset \mathfrak{B}_1 \supset \cdots \supset \mathfrak{B}_m = \mathfrak{B}$$

are the two chains, then one compares them to the two chains

$$\mathfrak{A} \supset \mathfrak{A}_1 \supset (\mathfrak{A}_1, \mathfrak{B}_1) \supset \cdots \supset \mathfrak{B}$$
$$\mathfrak{A} \supset \mathfrak{B}_1 \supset (\mathfrak{A}_1, \mathfrak{B}_1) \supset \cdots \supset \mathfrak{B}$$

for which the theorem is true according to Theorem 6 and the induction assumption.

One can also prove the theorem of Jordan-Hölder in the stronger form of a refinement theorem corresponding to the theorem of Schreier-Zassenhaus for groups. This requires the proof of a lemma analogous to the lemma of Zassenhaus. We shall omit these considerations here since they do not add materially to the theory of multigroups.

4. Strong normality. In the preceding we have used a definition of normality which is as weak as possible, but still sufficient to prove the theorem

of Jordan-Hölder. There are however other stronger types of normality with interesting properties.

One slight strengthening of our definition of normality is the following: The sub-multigroup $\mathfrak{A}$ shall be said to be normal in $\mathfrak{M}$ when any product $a_1 \cdot m$ can be written in the form $m \cdot a_2$ and conversely.

One can prove a few more properties of normal sub-multigroups on the basis of this definition. A more important type of normality may be defined as follows:

Strong normality. A sub-multigroup $\mathfrak{A}$ of $\mathfrak{M}$ is *strongly normal* when it contains left and right unit elements of $\mathfrak{M}$ and satisfies the following condition: To each m_1 there exists some left inverse m_1^{-1} such that

$$m_1 \mathfrak{A} m_1^{-1} \subset \mathfrak{A} \tag{6}$$

and to each m_2 some right inverse m_2^{-1} such that

$$m_2^{-1} \cdot \mathfrak{A} \cdot m_2 \subset \mathfrak{A} \tag{7}$$

Through multiplication of (6) and (7) by m_1 and m_2 one finds

$$m\mathfrak{A} = \mathfrak{A}m \tag{8}$$

for all m, so that strong normality implies normality in the sense we have used in the preceding. We can now prove:

THEOREM 8. *Strong normality implies reversibility.*

Proof. We shall show first that a strongly normal sub-multigroup $\mathfrak{A}$ is closed. This requires that any x and y such that

$$x \cdot a_1 \supset a_2, \qquad a_1 \cdot y \supset a_2 \tag{9}$$

belong to $\mathfrak{A}$. On account of the normality (8) it is sufficient to show that the second relation (9) implies that y belongs to $\mathfrak{A}$. According to (6) there exists a left inverse y^{-1} of y such that

$$y \cdot y^{-1} \subset \mathfrak{A}. \tag{10}$$

From the second relation in (9) follows

$$\mathfrak{A} \supset a_1 \cdot y \cdot y^{-1} \supset a_2 \cdot y^{-1}.$$

When this relation is multiplied by a left inverse of a_2 it follows that y^{-1} belongs to $\mathfrak{A}$. From (10) one concludes in the same manner that y belongs to $\mathfrak{A}$.

To prove the reversibility of $\mathfrak{A}$ we shall have to show that a relation

$$a_1 \cdot m_1 \supset m_2 \tag{11}$$

implies the existence of an a_2 in $\mathfrak{A}$ such that

$$m_1 \subset a_2 \cdot m_2.$$

In any case we can find an element z such that

$$z \cdot m_2 \supset m_1$$

and this implies according to (11) that

$$a_1 z \cdot m_2 \supset m_2.$$

When this relation is multiplied by a suitable left inverse of m_2 we find as in (10)

$$a_1 \cdot z \cdot \mathfrak{A} \supset \mathfrak{A}$$

and from the closure of $\mathfrak{A}$ one easily concludes that z belongs to $\mathfrak{A}$.

We can now prove further:

THEOREM 9. *All right and left inverses of an element m belong to the same co-set with respect to a strongly normal sub-multigroup, $\mathfrak{A}$.*

Proof. Let $\bar{m}$ be any left inverse of m and m^{-1} the left inverse for which

$$m \cdot m^{-1} \subset \mathfrak{A}.$$

Then

$$\bar{m} \cdot m \cdot m^{-1} \subset \bar{m}\mathfrak{A}$$

or

$$m^{-1} \subset \bar{m}\mathfrak{A}.$$

This shows that all left inverses of m belong to the same co-set $m^{-1} \cdot \mathfrak{A}$. But the right inverses of m must also belong to this co-set since

$$m \cdot m^{-1} \supset a_1$$

hence a suitable inverse a_2 of a gives

$$m \cdot m^{-1} \cdot a_2 \supset e$$

so that $m^{-1}a_2$ contains a right inverse of m.

Theorem 9 implies:

THEOREM 10. *The conditions of strong normality are equivalent to the relation*

$$m\mathfrak{A}m^{-1} = \mathfrak{A} \tag{12}$$

to hold for all m and all inverses m^{-1}.

It also shows that

$$m_1^{-1} \cdot m_1 \subset \mathfrak{A}, \qquad m_2 \cdot m_2^{-1} \subset \mathfrak{A}$$

for all inverses m_1^{-1} and m_2^{-1}. Furthermore if

$$m_1 x \supset m_2, \qquad y m_1 \supset m_2$$

then one finds that any other solutions of these relations belong to the co-sets $x\mathfrak{A}$ and $y\mathfrak{A}$ respectively.

5. Quotient groups. The preceding investigations on strong normality enables us to prove certain characteristic properties of strongly normal sub-multigroups. The following theorem is of great interest:

THEOREM 11. *When $\mathfrak{A}$ is strongly normal in $\mathfrak{M}$ then the quotient multigroup $\mathfrak{M}/\mathfrak{A}$ is an ordinary group.*

Proof. According to Theorem 1 the quotient multigroup $\mathfrak{M}/\mathfrak{A}$ has the absolute unit element $\mathfrak{A}$. Theorem 9 shows furthermore that

$$m \cdot \mathfrak{A} \cdot m^{-1} \cdot \mathfrak{A} = \mathfrak{A}.$$

This relation shows according to Theorem 2, chap. 1 that $m\mathfrak{A}$ is a left scalar in $\mathfrak{M}/\mathfrak{A}$. In the same manner follows that $m\mathfrak{A}$ is a right scalar hence every element of $\mathfrak{M}/\mathfrak{A}$ is a scalar and the quotient multigroup is a group.

The following theorem supplements Theorem 11:

THEOREM 12. *The necessary and sufficient condition that a multigroup $\mathfrak{M}$ be homomorphic to a group G is that $\mathfrak{M}$ contain a strongly normal sub-multigroup $\mathfrak{A}$ such that the group $\mathfrak{M}/\mathfrak{A}$ is isomorphic to G.*

Proof. Let us observe first that the theorem only assumes homomorphism in the weakest sense, but that the proof shows that $\mathfrak{M}$ must be strongly homomorphic to G. Let e^* be the unit element of G. According to Theorem 12, chap. 2 all those elements of $\mathfrak{M}$ which correspond to e^* form a closed sub-multigroup $\mathfrak{A}$ of $\mathfrak{M}$. Next let m_1 and m_2 be two elements with the same image m^* in G. We determine x and y such that

$$x \cdot m_1 \supset m_2, \qquad m_1 \cdot y \supset m_2$$

and find

$$x^* \cdot m^* = m^*, \qquad m^* \cdot y^* = m^*$$

or

$$x^* = y^* = e^*$$

hence x and y belong to $\mathfrak{A}$. This shows that $\mathfrak{A}$ is reversible and normal and all those elements which have the same image belong to the same co-set $m \cdot \mathfrak{A}$. One also finds that all inverses of m have the same image and $m \cdot m^{-1} \subset \mathfrak{A}$. This shows that $\mathfrak{A}$ is strongly normal. Finally the correspondence between $\mathfrak{M}$ and $\mathfrak{M}/\mathfrak{A}$ is a strong homomorphism.

From the fact that $\mathfrak{M}/\mathfrak{A}$ is a group one easily concludes: Every multigroup $\mathfrak{B}$ containing the strongly normal sub-multigroup $\mathfrak{A}$ of $\mathfrak{M}$ is reversible. When $\mathfrak{B}$ is normal in $\mathfrak{M}$ it is also strongly normal. Each $\mathfrak{B}$ corresponds to a sub-group of $\mathfrak{M}/\mathfrak{A}$ and to a normal subgroup if $\mathfrak{B}$ is normal in $\mathfrak{M}$. Conversely every sub-group of $\mathfrak{M}/\mathfrak{A}$ corresponds to a sub-multigroup $\mathfrak{B}$ of $\mathfrak{M}$ containing $\mathfrak{A}$, and a normal subgroup of $\mathfrak{M}/\mathfrak{A}$ corresponds to a strongly normal sub-multigroup $\mathfrak{B}$.

THEOREM 13. *The strongly normal sub-multigroups of $\mathfrak{M}$ form a Dedekind structure.*

Proof. If $\mathfrak{A}$ and $\mathfrak{B}$ are the strongly normal sub-multigroups then it is obvious that the union $[\mathfrak{A}, \mathfrak{B}] = \mathfrak{A} \cdot \mathfrak{B}$ is strongly normal. For the cross-cut $\mathfrak{D} = (\mathfrak{A}, \mathfrak{B})$ one finds that $m\mathfrak{D}m^{-1}$ belongs both to $\mathfrak{A}$ and to $\mathfrak{B}$, hence to $\mathfrak{D}$. From Theorem 5 follows that the Dedekind axiom is satisfied.

Let us say that a chain of multigroups in $\mathfrak{M}$ is a *strong composition series* when no further strongly normal terms can be intercalated. From the second law of isomorphism follows:

THEOREM 14. *Any two maximal strong composition series between two sub-multigroups $\mathfrak{A} \supset \mathfrak{B}$ have the same length and the simple quotient groups defined by consecutive terms in one chain are isomorphic to those of the other in some order.*

One can also extend this theorem by proving an analogue to the refinement theorem of Schreier-Zassenhaus for an arbitrary strong composition series.

We shall need the following theorem for certain applications:

THEOREM 15. *Let $\mathfrak{A}$ be a strongly normal sub-multigroup and an arbitrary sub-multigroup containing a unit element of $\mathfrak{M}$. Then the cross-cut $\mathfrak{D} = (\mathfrak{A}, \mathfrak{B})$ exists and is a strongly normal sub-multigroup of $\mathfrak{B}$ such that*

$$[\mathfrak{A}, \mathfrak{B}]/\mathfrak{A} \cong \mathfrak{B}/\mathfrak{D}.$$

Proof. The cross-cut $\mathfrak{D}$ is not void, because any b in $\mathfrak{B}$ must possess an inverse b^{-1} also contained in $\mathfrak{B}$, and $b \cdot b^{-1}$ belongs both to $\mathfrak{A}$ and to $\mathfrak{B}$. To show that $\mathfrak{D}$ is a sub-multigroup we shall have to show that the relations

$$d_1 \cdot x \supset d_2, \qquad y \cdot d_1 \supset d_2$$

have solutions in $\mathfrak{D}$, when d_1 and d_2 belong to $\mathfrak{D}$. There exist, however, solutions in $\mathfrak{B}$ and these solutions must also lie in $\mathfrak{A}$ since $\mathfrak{A}$ is closed. Furthermore the complexes

$$b\mathfrak{D}b^{-1}, \qquad b^{-1}\mathfrak{D}b$$

belong both to $\mathfrak{A}$ and to $\mathfrak{B}$, hence $\mathfrak{D}$ is strongly normal in $\mathfrak{B}$. Finally if

$$\mathfrak{B} = b_1\mathfrak{D} + b_2\mathfrak{D} + \cdots \tag{13}$$

is a co-set expansion of $\mathfrak{B}$ with respect to $\mathfrak{D}$, then

$$[\mathfrak{A}, \mathfrak{B}] = b_1\mathfrak{A} + b_2\mathfrak{A} + \cdots \tag{14}$$

is a co-set expansion of $[\mathfrak{A}, \mathfrak{B}]$ with respect to $\mathfrak{A}$. It follows namely that since every element of $[\mathfrak{A}, \mathfrak{B}]$ is contained in some product $a \cdot b$ every element of $[\mathfrak{A}, \mathfrak{B}]$ belongs to some co-set (14). If two co-sets in (14) were equal one would have a relation $b_i a \supset b_j$ and hence the relation

$$b_i x \supset b_j$$

would only have solutions x in $\mathfrak{A}$ according to a preceding remark. There is however always a solution in $\mathfrak{B}$, hence $b_i d \supset b_j$ where d belongs to $\mathfrak{D}$. This shows that b_i and b_j belong to the same co-set in (13) against assumption. The correspondence between the co-sets (13) and (14) is obviously an isomorphism.

6. Ultragroups. One finds that the cross-cut of an arbitrary set of strongly normal sub-multigroups is again strongly normal. The cross-cut of all strongly normal sub-multigroups of $\mathfrak{M}$ we shall call the *core* of $\mathfrak{M}$ and denote by $\mathfrak{C}$.

THEOREM 16. *The core $\mathfrak{C}$ is a unique minimal sub-multigroup of $\mathfrak{M}$ such that $\mathfrak{M}/\mathfrak{C}$ is a group.*

The core has the property that it contains all products $m \cdot m^{-1}$ and $m^{-1} \cdot m$ where m^{-1} is any inverse of m.

The core $\mathfrak{C}$ may itself contain a core $\mathfrak{C}_2$, the second core of $\mathfrak{M}$. This may contain a core $\mathfrak{C}_3$, etc. The quotient multigroups $\mathfrak{C}_{i-1}/\mathfrak{C}_i$ are all ordinary groups. It may happen that this sequence of cores exhausts the multigroup $\mathfrak{M}$

$$\mathfrak{M} \supset \mathfrak{C}_1 \supset \cdots \supset \mathfrak{C}_n \supset e \tag{15}$$

where e is a unit element. We shall say that the multigroup $\mathfrak{M}$ is an *ultragroup* when there exists some strong composition series

$$\mathfrak{M} \supset \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_m \supset e \tag{16}$$

exhausting $\mathfrak{M}$. Since all $\mathfrak{A}_i$ are closed, an ultragroup can contain only a single unit element.

THEOREM 17. *The necessary and sufficient condition that $\mathfrak{M}$ be an ultragroup is that it be exhausted by its chain of cores.*

Proof. The condition is obviously sufficient. To prove the necessity we shall have to assume that the descending chain condition holds for the strongly normal chains in $\mathfrak{M}$. Let us suppose that a strongly normal composition series (16) exists in $\mathfrak{M}$. If $\mathfrak{C}_n$ were the last core of $\mathfrak{M}$ then according to Theorem 15

$$\mathfrak{C}_n \supseteqq (\mathfrak{C}_n, \mathfrak{A}_1) \supseteqq (\mathfrak{C}_n, \mathfrak{A}_2) \supseteqq \cdots \supseteqq e$$

would be a further strong composition chain for $\mathfrak{C}_n$ against assumption.

The following example shows the existence of ultragroups which are not ordinary groups:

	e	a	b	c
e	e	a	b, c	b, c
a	a	e	b, c	b, c
b	b, c	b, c	e, a	e, a
c	b, c	b, c	e, a	e, a

Here e, a form a subgroup which is strongly normal in the whole multigroup and the quotient group is of order 2.

As in the case of groups the quotient groups $\mathfrak{A}_{i-1}/\mathfrak{A}_i$ do not determine $\mathfrak{M}$ uniquely. This leads to a more general formulation of the *Schreier extension problem,* namely to determine the ultragroups which correspond to a given set of quotient groups.

To conclude we shall prove:

THEOREM 18. *Every sub-multigroup containing the unit element of an ultragroup is itself an ultragroup.*

Proof. If the ultragroup $\mathfrak{M}$ has the strong decomposition series (16) then one finds according to Theorem 15, that

$$\mathfrak{B} \supseteqq (\mathfrak{B}, \mathfrak{A}_1) \supseteqq (\mathfrak{B}, \mathfrak{A}_2) \supseteqq \cdots \supseteqq e$$

is a strong composition series for $\mathfrak{B}$.

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THE GENERAL RATIONAL SOLUTION OF THE EQUATION $ax^2 - by^2 = z^3$.*

By E. FOGELS.

The diophantine equation

$$(1) \qquad ax^2 - by^2 = z^3$$

was treated by Euler, Lagrange, Legendre and other authors.[1] Euler assumed that $x\sqrt{a} + y\sqrt{b} = (u\sqrt{a} + v\sqrt{b})^3$, and found the two-parameter solution

$$(2) \qquad x = au^3 + 3buv^2 \quad y = 3au^2v + bv^3, \quad z = au^2 - bv^2.$$

He noted that (2) is not the general solution. For instance, the solution $x = 4$, $y = 1$, $z = 3$ of the equation $2x^2 - 5y^2 = z^3$ is not given by (2) with rational parameters u and v. Consequently there are *irrational* numbers u, v that give rational x, y, z.

We suppose that (x_0, y_0, z_0) is a rational solution of (1). Solving in algebraic numbers u, v the system

$$(3) \qquad y_0 = 3au^2v + bv^3, \qquad z_0 = au^2 - bv^2,$$

we have

$$a(au^3 + 3buv^2)^2 = (au^2 - bv^2)^3 + b(3au^2v + bv^3)^2$$
$$= z_0^3 + by_0^2 = ax_0^2,$$

or

$$(4) \qquad au^3 + 3buv^2 = \pm x_0.$$

From (3) and (4) we get the equations

$$(5) \qquad 4au^3 - 3z_0u - x_0 = 0, \qquad 4bv^3 + 3z_0v - y_0 = 0.$$

Here u and v are algebraic numbers of degree ≤ 3, in case x_0, y_0 and z_0 are rational. If one of these equations is reducible, then the other is too, because we have $z_0 = au^2 - bv^2$. Thus we shall consider only the case where both equations (5) are irreducible, or u and v are algebraic numbers of degree 3.

* Received by the Editors, September 5, 1937.

[1] L. E. Dickson, *History of the Theory of Numbers*, II, New York, 1934, pp. 531-539. See also L. E. Dickson, *Introduction to the Theory of Numbers*, German translation, 1931, p. 93.

A simple consideration shows that the algebraic domains $R(u)$ and $R(v)$ are identical.[2] Therefore we have the relations

$$(6) \qquad u^3 = pu + q$$

$$(7) \qquad v = \alpha + \beta u + \gamma u^2,$$

where $p, q, \alpha, \beta, \gamma$ are rational coefficients.

The product of two numbers of the domain $R(u)$ is

$$(8) \qquad (\alpha + \beta u + \gamma u^2)(\mathrm{A} + \mathrm{B}u + \Gamma u^2)$$
$$= \alpha\mathrm{A} + (\beta\Gamma + \gamma\mathrm{B})q + \{\alpha\mathrm{B} + \beta\mathrm{A} + (\beta\Gamma + \gamma\mathrm{B})p + \gamma\Gamma q\}u$$
$$+ \{\alpha\Gamma + \beta\mathrm{B} + \gamma\mathrm{A} + \gamma\Gamma p\}u^2;$$

hence we have the following: If the algebraic number v is defined by (7) and if

$$v^2 = \mathrm{A} + \mathrm{B}u + \Gamma u^2,$$

then

$$(9) \quad \mathrm{A} = \alpha^2 + 2q\beta\gamma, \quad \mathrm{B} = 2\alpha\beta + 2p\beta\gamma + q\gamma^2, \quad \Gamma = \beta^2 + 2\alpha\gamma + p\gamma^2.$$

With these results we obtain from (8) the number v^3. Substituting the values of v, v^2, v^3 in (2) and using (6) we express the rational numbers x, y, z in the form $a_0 + a_1 u + a_2 u^2$, where each of the coefficients a_1 and a_2 must vanish. Thus from the equation $z = -b\mathrm{A} - b\mathrm{B}u + (a - b\Gamma)u^2$ we get

$$(10) \qquad z = -b\mathrm{A}, \qquad \mathrm{B} = 0, \qquad \Gamma = a/b.$$

Further, from the equation $x = 4aq + (4ap + 3b\mathrm{A})u$ we get

$$(11) \qquad x = 4aq, \qquad \mathrm{A} = -(4ap/3b).$$

Finally, the equation

$$y = \tfrac{4}{3}a(3q\beta - p\alpha) + \tfrac{4}{3}a(2p\beta + 3q\gamma)u + \tfrac{4}{3}a(3\alpha + 2p\gamma)u^2$$

gives

$$(12) \quad y = \tfrac{4}{3}a(3q\beta - p\alpha), \qquad 3\alpha + 2p\gamma = 0, \qquad 2p\beta + 3q\gamma = 0.$$

Thus we get five relations between the coefficients $p, q, \alpha, \beta, \gamma$. Three of these relations are independent, e. g.

$$(13) \quad \alpha = -\tfrac{2}{3}p\gamma, \qquad q\gamma = -\tfrac{2}{3}p\beta, \qquad 3b\beta^2 - bp\gamma^2 - 3a = 0.$$

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[2] From equations (5) we get $4au^2 = 3z_0 + x_0/u$, $4v^2 = -3z_0 + y_0/v$. Therefore we have $4z_0 = 4au^2 - 4bv^2 = 6z_0 + x_0/u - y_0/v$ or $x_0/u = y_0/v - 2z_0$. Hence the domains $R(1/u) = R(u)$ and $R(1/v) = R(v)$ are identical.

Supposing $\gamma \neq 0$ [3] we find $p = 3(b\beta^2 - a)/b\gamma^2$, $q = -2\beta(b\beta^2 - a)/b\gamma^3$. Then from the equations (10)-(13) we see that

$$x = -8a\beta(b\beta^2 - a)/b\gamma^3, \quad y = -8a^2(b\beta^2 - a)/b^2\gamma^3, \quad z = 4a(b\beta^2 - a)/b\gamma^2.$$

Writing

$$(14) \qquad b\beta \equiv s, \qquad -(2/b\gamma) \equiv t,$$

we get the solution

$$(15) \quad x = as(s^2 - ab)t^3, \qquad y = a^2(s^2 - ab)t^3, \qquad z = a(s^2 - ab)t^2,$$

where s and t are independent rational parameters.

If for any s the equation (6) or the equation

$$(16) \qquad \xi^3 = 3(s^2 - ab)\xi - 2s(s^2 - ab) \qquad (\xi = b\gamma u)$$

is reducible, then (15) gives the same rational solution as (2). But in all cases when $ab \neq 0$, there is an infinity of rational numbers for which (16) is irreducible. In the corresponding solutions the parameters u, v in (2) are algebraic numbers of degree 3.

It follows that (15) is the general rational solution of the equation (1).

From the equations (15) we have $s = ax/y$. Consequently the equation (16) is reducible for all numbers of the form

$$(17) \qquad s = a\frac{au^3 + 3buv^2}{3au^2v + bv^3}$$

where u and v are rational, and conversely: if the equation is reducible, then s can be represented in the form (17).

In Euler's example the equation $2x^2 - 5y^2 = z^3$ has the solution $x = 4$, $y = 1$, $z = 3$: $s = 8$ and the equation (16) is irreducible. We get the solution from (15) by taking $t = 1/6$.

Note. In the preliminary investigation it was not required that a and b should be integers and "rationality" was not restricted. Therefore the results found are useful in any domain of rationality R, if we suppose that the coefficients a, b, the parameters u, v (resp. s, t) and the solutions x, y, z are numbers of the domain R.

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[3] The case $\gamma = 0$ corresponds to the equation $a(x^2 - y^2) = z^3$ that we shall not consider.

THE DISTRIBUTION OF INTEGERS REPRESENTED BY QUADRATIC FORMS.*

By R. D. James.

1. In a paper published some time ago, Landau [1] proved the following result.

Let $B(x)$ denote the number of positive integers $m \leqq x$ which may be represented in the form $m = u^2 + v^2$, where u and v are integers. Then

$$\lim_{x \to \infty} x^{-1} \sqrt{\log x}\, B(x) = b,$$

where b is a positive constant.

The object of the present paper is to show that a similar result holds for the integers represented by a binary quadratic form of discriminant $d \leqq -3$. We consider only primitive integral forms $au^2 + buv + cv^2$ of discriminant $d = b^2 - 4ac$, in which a, b, and c are integers having no common factor > 1. An integer m is said to be represented by such a form if there exist integers r and s such that

$$m = ar^2 + brs + cs^2.$$

THEOREM. *Let $B(x)$ denote the number of positive integers $m \leqq x$ which are prime to d and which are represented by some primitive integral form of discriminant $d \leqq -3$. Then* [2]

$$B(x) - bx/\sqrt{\log x} = O(x/\log x),$$

where b is a positive constant. The explicit value of b is given in Section 5.

This theorem does not include the Landau result because of the assumption that m is prime to d. For any particular numerical value of d this restriction could be removed, but it does not seem possible to do so if a general statement is desired.

2. In this section we examine the form of the prime factors of the integers m which are represented by $au^2 + buv + cv^2$. We make use of the Kronecker symbol $(d|p)$ when p is a prime. It is defined as follows. If p is an odd prime,

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[1] E. Landau, "Über die Einteilung der positiven ganzen Zahlen in vier Klassen," *Archiv der Mathematik und Physik* (3), vol. 13 (1908), pp. 305-312.

[2] We write $f(x) = O(g(x))$ if for $x > x_0$ we have $|f(x)| < Kg(x)$, where K is a positive constant.

$$(d|p) = \begin{cases} 1, & \text{if } d \text{ is a quadratic residue of } p, \\ 0, & \text{if } p|d, \\ -1, & \text{if } d \text{ is a non-residue of } p. \end{cases}$$

If $p = 2$ and $d \equiv 0$ or $1 \pmod 4$,

$$(d|2) = \begin{cases} 1, & \text{if } d \equiv 1 \pmod 8, \\ 0, & \text{if } 2|d, \\ -1, & \text{if } d \equiv 5 \pmod 8. \end{cases}$$

In our case $d = b^2 - 4ac$ and hence the condition $d \equiv 0$ or $1 \pmod 4$ is satisfied.

LEMMA 1. *Let p be any prime for which $(d|p) = -1$. If $m = p^{2t+1}M$, where $p \nmid M$, then m is not represented by any primitive form of discriminant d.*

Proof. We use the notation (a, b, c) for the form $au^2 + buv + cv^2$. We may assume that $p \nmid a$. For, if $p|a$ but $p \nmid c$, the transformation $u = -V$, $v = U$ transforms (a, b, c) into the equivalent form $(c, -b, a)$, and equivalent forms represent the same integers. If $p|a$ and $p|c$, then $p \nmid b$, since the form is primitive. Then the transformation $u = U$, $v = U + V$ transforms (a, b, c) into the equivalent form $(a + b + c,\ b + 2c,\ c)$ in which $p \nmid (a + b + c)$.

Suppose now that $m = p^{2t+1}M$ is represented by (a, b, c) with $p \nmid a$. We shall show that this supposition leads to a contradiction. We have

$$\begin{aligned} m = p^{2t+1}M &= ar^2 + brs + cs^2, \\ 4ap^{2t+1}M &= (2ar + bs)^2 - ds^2, \end{aligned} \tag{2.11}$$

so that

$$(2ar + bs)^2 \equiv ds^2 \pmod{4p^{2t+1}}. \tag{2.12}$$

We now distinguish two cases.

Case I. Let p be an odd prime. Since $(d|p) = -1$ it follows from (2.12) that $p|s$, $p|(2ar + bs)$. Since $p \nmid a$ this means that $p|r$, and from (2.11) that $p^2|m$. Thus we have

$$p^{2t-1}M = ar_1^2 + br_1s_1 + cs_1^2,$$

where $r = r_1p$, $s = s_1p$. Repetition of this argument leads to the equation

$$pM = ar_t^2 + br_ts_t + cs_t^2. \tag{2.13}$$

As before we have

$$(2ar_t + bs_t)^2 \equiv ds_t^2 \pmod{4p}, \tag{2.14}$$

and from (2.14) and (2.13) we find that $p|r_t$, $p|s_t$, $p^2|pM$. This is a contradiction since $p \nmid M$.

Case II. Let $p = 2$. Then the congruence (2.12) becomes

$$(2ar + bs)^2 \equiv ds^2 (\mathrm{mod}\ 2^{2j+3}),$$

When $(d|2) = -1$ we have $d \equiv 5 \pmod 8$. Hence

$$(2.15) \qquad (2ar + bs)^2 \equiv 5s^2 \pmod 8.$$

Since d is odd, b must be odd and it is easily seen that r and s in (2.15) must both be even. We therefore have

$$2^{2t-1}M = ar_1^2 + br_1s_1 + cs_1^2.$$

Proceeding as in Case I, we finally obtain

$$2M = ar_t^2 + br_ts_t + cs_t^2,$$
$$(2ar_t + bs_t)^2 \equiv 5s_t^2 \pmod 8.$$

As before r_t and s_t are both even so that $4|2M$. This contradicts $2 \nmid M$.

LEMMA 2. *Let m be any integer prime to d. Write $m = MN^2$, where M has no square factor > 1. If no prime p for which $(d|p) = -1$ divides M, then m is represented by some primitive form of discriminant d.*

Proof. It is well known [3] that the number of representations of m by a representative system of primitive forms of discriminant d is given by

$$(2.21) \qquad \psi(m) = w \sum_{g|N} \prod_{p|MN^2g^{-2}} \{1 + (d|p)\},$$

where w is the number of automorphs of the form. Each product is positive or zero and hence the right side of (2.21) is greater than or equal to the term in which $g = N$. Thus

$$(2.22) \qquad \psi(m) \geqq 2 \prod_{p|M} \{1 + (d|p)\}.$$

No prime p for which $(d|p) = -1$ divides M. Therefore each term of the product (2.22) is equal to 2. This shows that $\psi(m) > 0$.

3. In this section we introduce the Dirichlet series associated with the function $B(x)$. Let $b(n)$ be defined as follows.

$$b(n) = \begin{cases} 1, & \text{if } n \text{ is prime to } d \text{ and represented} \\ & \quad \text{by some form of discriminant } d, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$B(x) = \sum_{1 \leqq n \leqq x} b(n).$$

[3] L. E. Dickson, *Introduction to the Theory of Numbers*, Chicago (1929), Section 50.

If $s = \sigma + ti$ is a complex variable, the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} b(n) n^{-s} \tag{3.1}$$

is absolutely convergent for $\sigma > 1$.

Throughout this section and in Section 5, p will always denote a prime for which $(d|p) = +1$, and q will always denote a prime for which $(d|q) = -1$.

LEMMA 3. *For $\sigma > 1$ we have*

$$f(s) = \prod_p (1 - p^{-s})^{-1} \prod_q (1 - q^{-2s})^{-1}.$$

Proof. The infinite product is absolutely convergent since the absolute value of the infinite series Σp^{-s} is less than $\Sigma n^{-\sigma}$, which is a convergent series. Using Lemmas 1 and 2 we have

$$\begin{aligned}
&\prod_{p \leqq N} (1 - p^{-s})^{-1} \prod_{q \leqq N} (1 - q^{-2s})^{-1} \\
&\quad = \prod_{p \leqq N} (1 + p^{-s} + p^{-2s} + \cdots) \prod_{q \leqq N} (1 + q^{-2s} + q^{-4s} + \cdots) \\
&\quad = \Sigma' b(n) n^{-s},
\end{aligned}$$

where the prime indicates that the summation is over all integers n whose prime factors are $\leqq N$. Also,

$$\Sigma' b(n) n^{-s} = \sum_{n=1}^{N} b(n) n^{-s} + \sum_{n > N}{}' b(n) n^{-s}.$$

Then

$$\begin{aligned}
&\Big| \prod_{p \leqq N} (1 - p^{-s})^{-1} \prod_{q \leqq N} (1 - q^{-2s})^{-1} - \sum_{n=1}^{N} b(n) n^{-s} \Big| \\
&\quad = \Big| \sum_{n > N}{}' b(n) n^{-s} \Big| \leqq \sum_{n=N+1}^{\infty} n^{-\sigma} \\
&\quad \leqq (N+1)^{-\sigma} + \int_{N+1}^{\infty} u^{-\sigma} du = (N+1)^{-\sigma} + (\sigma - 1)^{-1} (N+1)^{\sigma - 1}.
\end{aligned}$$

Hence

$$\lim_{N \to \infty} \Big\{ \prod_{p \leqq N} (1 - p^{-s})^{-1} \prod_{q \leqq N} (1 - q^{-2s})^{-1} - \sum_{n=1}^{N} b(n) n^{-s} \Big\} = 0.$$

Moreover,

$$\lim_{N \to \infty} \prod_{p \leqq N} (1 - p^{-s})^{-1} \prod_{q \leqq N} (1 - q^{-2s})^{-1}$$

and

$$\lim_{N \to \infty} \sum_{n=1}^{N} b(n) n^{-s}$$

exist and therefore are equal.

The next step is to express $f^2(s)$ as the product of two Dirichlet L-functions. Let $\Delta = -d$ and let

$$\chi_0(n) = \begin{cases} 1, \text{ if } (n, \Delta) = 1, \\ 0, \text{ otherwise}; \end{cases}$$

$$\chi_1(n) = \prod_{i=1}^{k} (d \mid p_i), \text{ when } n = \prod_{i=1}^{k} p_i.$$

Then $\chi_0(n)$ and $\chi_1(n)$ are characters[4] modulo Δ. For $\sigma > 1$ let

$$L(s, \chi_j) = \sum_{n=1}^{\infty} \chi_j(n) n^{-s}, \qquad (j = 0, 1).$$

LEMMA 4. *For $\sigma > 1$ we have*

$$f^2(s) = g(s) L(s, \chi_0) L(s, \chi_1),$$

where

$$g(s) = \prod_q (1 - q^{-2s})^{-1}.$$

Proof. By the argument used in the proof of Lemma 3 it can be shown that

$$L(s, \chi_j) = \prod_p (1 - \chi_j(p) p^{-s})^{-1} \prod_q (1 - \chi_j(q) q^{-s})^{-1}.$$

Hence

$$L(s, \chi_0) L(s, \chi_1) = \prod_p (1 - p^{-s})^{-2} \prod_q (1 + q^{-s})^{-1} (1 - q^{-s})^{-1}.$$

The desired result now follows from Lemma 3.

LEMMA 5. *The function $f(s)$ defined by (3.1) for $\sigma > 1$ and elsewhere by analytic continuation has the following properties. There exists a constant $c > 1$ such that*

(I) $|f(s)| < (\log|t|)^c$, *for* $\sigma \geqq 1 - (\log|t|)^{-c}$, $|t| \geqq 3$.

(II) *In the domain* $\sigma \geqq 1 - (\log|t|)^{-c}$, $|t| \geqq 3$; $\sigma \geqq 1 - (\log 3)^{-c}$, $|t| \leqq 3$; *with a cut from* $s = 1 - (\log 3)^{-c}$ *to* $s = 1$ *the function* $f(s)$ *is regular.*

(III) *For* $|s - 1| \leqq (\log 3)^{-c}$ *the function* $(s-1)s^{-2}f^2(s)$ *may be expanded in the absolutely convergent series*

$$(s-1)s^{-2}f^2(s) = a + \sum_{j=1}^{\infty} a_j (s-1)^j,$$

where $a > 0$ is given by

$$a = \{\prod_q (1 - q^{-2})^{-1}\} \Delta^{-1} \phi(\Delta) L(1, \chi_1). \tag{3.2}$$

Proof. The properties of the L-functions which we use here are to be found in Landau's *Handbuch der Lehre von der Verteilung der Primzahlen.*

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[4] See, for example, E. Landau, *Vorlesungen über Zahlentheorie,* I, Leipzig (1927), pp. 83-87.

(I). For $\sigma \geqq 1 - (\log |t|)^{-1}$, $|t| \geqq 3$ we have

$$|L(s,\chi_j)| < A_1 \log |t|,$$
$$|g(s)| < A_2.$$

Hence

$$\begin{aligned}|f^2(s)| &< A_1^2 A_2 (\log |t|)^2 \\ &< (\log |t|)^{2n},\end{aligned}$$

provided n is chosen so that $(\log 3)^{2n-2} \geqq A_1^2 A_2$.

(II). The function $L(s,\chi_0)$ is regular in the given domain except for a simple pole with residue $\Delta^{-1}\phi(\Delta)$ at $s = 1$. The functions $g(s)$ and $L(s,\chi_1)$ are regular throughout the domain. Since $s = 1$ is excluded by the cut, it follows from Lemma 4 that $f^2(s)$ is a regular function. Finally, because of the cut, the function $f(s)$ itself is regular in the given domain.

(III). The functions $s^{-2}g(s)$ and $L(s,\chi_1)$ are regular at $s = 1$. The function $L(s,\chi_0)$ has a simple pole with residue $\Delta^{-1}\phi(\Delta)$ at $s = 1$. Hence for $|s - 1| \leqq \delta$ we have

$$s^{-2}g(s) = g(1) + \sum_{j=1}^{\infty} c_j (s-1)^j,$$
$$L(s,\chi_1) = L(1,\chi_1) + \sum_{j=1}^{\infty} d_j (s-1)^j,$$
$$(s-1)L(s,\chi_0) = \Delta^{-1}\phi(\Delta) + \sum_{j=1}^{\infty} e_j (s-1)^j.$$

From these equations it follows that

$$s^{-2}(s-1)f^2(s) = g(1)L(1,\chi_1)\Delta^{-1}\phi(\Delta) + \sum_{j=1}^{\infty} h_j (s-1)^j.$$

We now choose c so that

$$(\log 3)^{2c-2} \geqq A_1^2 A_2,$$
$$(\log 3)^{-c} \leqq \delta.$$

This completes the proof.

4. Before proceeding with the proof of the main theorem we state and prove two lemmas.

LEMMA 6. *If* $\delta = 1/\log x$, $x \geqq e^2$, $\tfrac{1}{2} \leqq s \leqq \tfrac{3}{2}$, we have

$$\{(1+\delta)^s - 1\}\{\log(1+\delta)\}^{-1} = s + O(1/\log x).$$

Proof. Using Taylor's theorem we have

$$(1+\delta)^s = 1 + \delta s + \frac{\delta^2}{2} s(s-1)\xi^{s-2}, \qquad 1 < \xi < 1 + \delta;$$

$$\log(1+\delta) = \delta - \frac{\delta^2}{2}\eta^{-2}, \qquad 1 < \eta < 1 + \delta.$$

Hence

$$\delta s - 2\delta^2 < (1+\delta)^s - 1 < \delta s + 2\delta^2,$$
$$\delta - \delta^2 < \log(1+\delta) < \delta + \delta^2,$$
$$(s-2\delta)(1+\delta)^{-1} < \{(1+\delta)^s - 1\}\{\log(1+\delta)\}^{-1} < (s+2\delta)(1-\delta)^{-1},$$
$$s - 7\delta < \{(1+\delta)^s - 1\}\{\log(1+\delta)\}^{-1} < s + 7\delta.$$

The statement of the lemma follows from the last inequality since $\delta = 1/\log x$.

LEMMA 7. *Let* $\theta = 1 - (\log 3)^{-c}$. *Then*

$$\int_\theta^1 x^s(1-s)^{-\frac{1}{2}}ds = \sqrt{\pi}\, x/\sqrt{\log x} + O(x/\log x).$$

Proof. We have

$$(4.1) \quad \int_\theta^1 x^s(1-s)^{-\frac{1}{2}}ds = \int_0^1 x^s(1-s)^{-\frac{1}{2}}ds + O\{\int_0^\theta x^s(1-s)^{-\frac{1}{2}}ds\}$$
$$= \int_0^1 x^s(1-s)^{-\frac{1}{2}}ds + O(x^\theta).$$

Now let $1 - s = v/\log x$, so that $ds = -dv/\log x$ and $x^s = xe^{-v}$. Then

$$(4.2) \quad \int_0^1 x^s(1-s)^{-\frac{1}{2}}ds = (x/\sqrt{\log x})\int_0^{\log x} e^{-v}v^{-\frac{1}{2}}dv.$$

Next, we have

$$(4.3) \quad \int_0^{\log x} e^{-v}v^{-\frac{1}{2}}dv = \int_0^\infty e^{-v}v^{-\frac{1}{2}}dv + O\{\int_{\log x}^\infty e^{-v}v^{-\frac{1}{2}}dv\}$$
$$= \Gamma(\tfrac{1}{2}) + O\{(1/\sqrt{\log x})\int_{\log x}^\infty e^{-v}dv\}$$
$$= \sqrt{\pi} + O(1/\sqrt{\log x}).$$

From (4.1), (4.2), and (4.3) we obtain

$$\int_\theta^1 x^s(1-s)^{-\frac{1}{2}}ds = \sqrt{\pi}\, x/\sqrt{\log x} + O(x/\log x) + O(x^\theta).$$

This proves the lemma since $x^\theta = O(x/\log x)$.

5. We are now in a position to prove the main theorem. The conditions satisfied by our function are precisely those satisfied by the function $f(s)$ used by Landau in the paper referred to in Section 1. We may therefore use his equation (8), page 311. We have

$$(5.1) \quad \sum_{1 \leq n \leq x} b(n)\log(x/n) = (1/\pi i)\int_\theta^1 s^{-2}x^s f(s)\,ds + O(x\exp(-\sqrt[3c]{\log x})).$$

In this equation replace x by $x + \delta x$, where $\delta = 1/\log x$. From this new equation subtract equation (5.1). This gives

$$\log(1+\delta)\sum_{1\leqq n\leqq x} b(n) = (1/\pi i)\int_\theta^1 \{(1+\delta)^s - 1\} s^{-2} x^s f(s)\,ds$$
$$- \sum_{x<n\leqq x+\delta x} b(n)\log((x+\delta x)/n) + O(x\exp(-\sqrt[3c]{\log x})).$$

Now

$$\sum_{x<n\leqq x+\delta x} b(n)\log((x+\delta x)/n) = O(\delta x\log(1+\delta))$$

and hence

$$(5.2)\quad B(x) = (1/\pi i)\int_\theta^1 \{(1+\delta)^s - 1\}\{\log(1+\delta)\}^{-1} s^{-2} x^s f(s)\,ds + O(\delta x).$$

The proof is completed in three steps.

First Step. In the integral in (5.2) replace $s^{-2}f(s)$ by $s^{-1}i\sqrt{a}\,(1-s)^{-\frac{1}{2}}$. By Lemma 5, (III) the error will be

$$O\{\int_\theta^1 \{(1+\delta)^s - 1\}\{\log(1+\delta)\}^{-1} x^s ds\}.$$

By Lemma 6 this is

$$O\{\int_\theta^1 s x^s ds\} = O\{\int_\theta^1 x^s ds\} = O\{[x^s/\log x]_\theta^1\} = O(x/\log x).$$

Since $\delta = 1/\log x$, (5.2) becomes

$$(5.3)\quad B(x) = (\sqrt{a}/\pi)\int_\theta^1 \{(1+\delta)^s - 1\}\{\log(1+\delta)\}^{-1} s^{-1}(1-s)^{-\frac{1}{2}} x^s ds$$
$$+ O(x/\log x).$$

Second Step. In the integral in (5.3) replace $\{(1+\delta)^s - 1\}\{\log(1+\delta)\}^{-1}$ by s. By Lemma 6 the error will be

$$O\{(1/\log x)\int_\theta^1 s^{-1}(1-s)^{-\frac{1}{2}} x^s ds\}.$$

By Lemma 7 this is $O(x/\log x)$. Then (5.3) becomes

$$(5.4)\qquad B(x) = (\sqrt{a}/\pi)\int_\theta^1 x^s(1-s)^{-\frac{1}{2}} ds + O(x/\log x).$$

Third Step. Replace the integral in (5.4) by its value from Lemma 7. This gives

$$B(x) = bx/\sqrt{\log x} + O(x/\log x),$$

where $b = \sqrt{a/\pi}$ and a is given by (3.2).

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ON THE REPRESENTATIONS OF THE SYMMETRIC GROUP.*

By G. de B. Robinson.

Introduction. In the study of the irreducible representations of the symmetric group two methods are available. The *first* is an application of the Frobenius-Schur theory of the characters which is valid for any group, the *second* is the 'substitutional analysis' of Young. Neither of these methods tells the whole story, and they should be used in conjunction.[1]

Here we propose to show that the two problems dealt with by Murnaghan [2] in his paper "On the Representations of the Symmetric Group," lend themselves to a treatment by Young's methods. An answer to the first problem may be obtained from a formula given by Young; [3] we shall clarify this somewhat embodying the result in the rule Y or Y'. Littlewood and Richardson [4] have given a theorem involving a rule LR which, if accompanied by a satisfactory proof, would provide an answer to Murnaghan's second problem. We propose to supply a proof, basing it directly on Y. The chief advantage of these methods is in the simplicity of the final expression of the result. It is unnecessary to refer to any tables, and the irreducible components appear explicitly,—no cancellation is necessary. On the other hand if we are concerned with the characters [5] the Frobenius-Schur theory is essential. In the last section of the paper we give illustrations of the application of these rules.

I must express my thanks to Mr. D. E. Littlewood for suggestions which led to the revision of the original draft of § 5 on lattice permutations. A specific acknowledgment is made in the text.

1. The product and power representations of the full linear group. The theory of the representations of the symmetric group S_n on n symbols is very closely associated with that of the rational representations of the full linear group L, whose degree we shall take to be l. There is an infinity of such irreducible representations but those of *order* n are to found amongst the irreducible components of the Kronecker product

$$\Pi_n(L) = L \times L \times L \times \cdots n \text{ factors}. \tag{1.1}$$

* Received February 4, 1938; Revised April 18, 1938.

[1] [20], chapter V.

[2] [5], p. 469 and p. 478; cf. also [8].

[3] [21], Part VI, p. 199.

[4] [3], p. 119.

[5] [6].

The degree of $\Pi_n(L)$, known as the *product* representation, is l^n. A very elegant reduction of $\Pi_n(L)$ has been given by Schur,[6] who shows that with every irreducible representation [7] (λ) of S_n of degree f_λ is associated an irreducible representation $T^{(\lambda)}(L)$, or as we shall write $\{\lambda\}$ of L. This reduction is accomplished by constructing matrices of degree l^n permutable with $\Pi_n(L)$, which interchange the n sets of variables [8] according to the permutations of S_n. These $n!$ matrices yield a representation of S_n, and from Schur's Lemma it follows that $\{\lambda\}$ appears in $\Pi_n(L)$ with multiplicity f_λ.

If we suppose that all the factors in (1.1) operate on the same set of variables the resulting representation is known as the n-th *power* representation of L and denoted $P_n(L)$. Corresponding to a given partition $(\alpha_1, \alpha_2, \cdots, \alpha_\nu)$ or (α) of n where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\nu$, we may construct the representation

$$P_{\alpha_1}(L) \times P_{\alpha_2}(L) \times \cdots \times P_{\alpha_\nu}(L).$$

Let us denote by P_α the sub-group of S_n of order $\alpha_1!\,\alpha_2!\cdots\alpha_\nu!$, which is the direct product of the sub-groups S_{α_1} on the first α_1 symbols, S_{α_2} on the next α_2 symbols, and so on. This sub-group gives rise to a permutation representation [9] of S_n of degree

$$\frac{n!}{\alpha_1!\,\alpha_2!\cdots\alpha_\nu!} = \binom{n}{\alpha},$$

which we may denote $\Delta(\alpha)$, extending the notation to write

$$(1.2) \qquad \Delta_{(\alpha)}(L) = P_{\alpha_1}(L) \times P_{\alpha_2}(L) \times \cdots \times P_{\alpha_\nu}(L).$$

In particular $\Pi_n(L) = \Delta_{(1^n)}(L)$. Evidently $\Delta_{(n)}(L) = P_n(L) = \{n\}$, so that we may write (1.2) in the form

$$(1.3) \qquad \Delta_{(\alpha)}(L) = \{\alpha_1\} \times \{\alpha_2\} \times \cdots \times \{\alpha_\nu\}.$$

Clearly also we may write

$$(1.4) \qquad \Delta_{(\alpha)}(L) = \Delta_{(\beta)}(L) \times \Delta_{(\gamma)}(L),$$

where the numbers $\beta_1, \beta_2, \cdots, \beta_\lambda;\ \gamma_1, \gamma_2, \cdots, \gamma_\mu$ are the α's possibly rearranged, so that $\beta_i \geq \beta_{i+1}$ and $\gamma_j \geq \gamma_{j+1}$ for all i, j, $\lambda + \mu = \nu$, $\sum_{i=1}^{\lambda} \beta_i = l$, $\sum_{j=1}^{\mu} \gamma_j = m$, and $l + m = n$.

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[6] [13], §§ 1 and 2.

[7] No confusion will result if we use the same symbol (λ) to denote the corresponding conjugate set of S_n.

[8] Cf. [19] and [1].

[9] [10].

By identifying the characters[10] *it follows that* $\{\lambda\}$ *appears in* $\Delta_{(a)}(L)$ *with the same multiplicity as does* (λ) *in* $\Delta(\alpha)$.

2. A formula of Young applied to the reduction of $\Delta(\alpha)$. A method for determining the irreducible components of $\Delta(\alpha)$, or of $\Delta_{(a)}(L)$, has been given by Murnaghan.[11] We shall make use of a formula [3] of Young's which is applicable to the situation:

$$\frac{1}{n!}\binom{n}{\alpha}\Gamma P_a = \Sigma[\Pi S_{rs}^{\lambda_{rs}}]\frac{n!}{f_a}T_a. \tag{2.1}$$

It is not necessary here to go into any detailed account of Young's analysis, we shall merely give enough to explain the symbols involved. Corresponding to a conjugate set (α) of S_n we may construct a *tableau*

$$[\alpha]:\quad \begin{matrix} a_{11}\, a_{12} \cdots a_{1\alpha_1} \\ a_{21}\, a_{22} \cdots a_{2\alpha_2} \\ \cdot \quad \cdots \\ a_{v1}\, a_{v2} \cdot a_{v\alpha_v} \end{matrix}$$

where, as before, $\alpha_i \geqq \alpha_{i+1}$ for all i and $\sum_{i=1}^{v}\alpha_i = n$. From the rows of $[\alpha]$ we construct substitutional expressions $\{a_{i1}\, a_{i2} \cdots a_{i\alpha_i}\}$ representing the sum of all the operations of S_{α_i}; multiplying these together we obtain

$$P_a = \{a_{11}\, a_{12} \cdots a_{1\alpha_1}\}\{a_{21}\, a_{22} \cdots a_{2\alpha_2}\} \cdots \{a_{v1} \cdots a_{v\alpha_v}\}.$$

The brackets $\{\ \}$ and their product P_a as well as other expressions N_a, T_a which we shall form are specially chosen members of the group-ring to which S_n gives rise. The members of this group-ring may be thought of as operators but we shall not stress this interpretation. The relation with our former P_a is so close we need not distinguish between them.

Similarly from the j-th column of $[\alpha]$ we construct $\{a_{1j}\, a_{2j} \cdots\}'$, where now every odd permutation has coefficient -1. Such an expression Young calls a 'negative symmetric group,' and from the columns we obtain

$$N_a = \{a_{11}\, a_{21} \cdots a_{v1}\}'\{a_{12}\, a_{22} \cdots\}' \cdots \{a_{1\alpha_1} \cdots\}'.$$

Thus with any given arrangement of the n letters in the tableau form $[\alpha]$ we may associate the product P_aN_a, and denoting summation over all possible $n!$ arrangements by Γ we may write

[10] [13]; [5], pp. 444-448; and [8], p. 45. We use $\Delta(a)$ with the same meaning as does Murnaghan, while our $\{\lambda\}$ has a different significance.

[11] [5]. The results are tabulated up to $n = 9$.

$$T_a = \left(\frac{f_a}{n!}\right)^2 \Gamma P_a N_a.$$

We may define the tableau $[\alpha]$ as *standard* if the letters in each row and column appear in the order of some given sequence. It can be shown that just f_a of the $n!$ are standard, where

$$f_a = n! \frac{\prod_{r,s} (\alpha_r - \alpha_s - r + s)}{\prod_r (\alpha_r + \nu - r)!},$$

and that T_a may be expressed in terms of them only:

$$T_a = \frac{f_a}{n!} (P_1 N'_1 + P_2 N'_2 + \cdots + P_{f_a} N'_{f_a}),$$

where $N'_r = N_r M_r$. The introduction of the multiplicative factor M_r is necessary to obtain orthogonality, but we need not go into this. For us the important fact is that corresponding to each conjugate set (α) we have a tableau $[\alpha]$ and a resulting T_a which leads directly to the irreducible representation [12] of S_n.

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[12] It may be worth while at this point to relate some recent work by Specht [14] and [16] with this analysis of Young. Following Schur [12], if we replace the symbols a_{ij}, in dictionary order, by x_i $(i = 1, 2, \cdots, n)$, we may uniquely associate with each tableau $[\alpha]$ the product of powers

$$\delta(x) = (x_1 x_2 \cdots x_{a_1})^0 (x_{a_1+1} x_{a_1+2} \cdots x_{a_1+a_2})^1 \cdots (x_{n-a_\nu+1} x_{n-a_\nu+2} \cdots x_n)^{\nu-1}.$$

A permutation P of S_n will leave $\delta(x)$ unaltered or transform it into $\delta^P(x)$ according as P is, or is not, contained in P_a. Instead of treating the group ring directly Specht constructs functions

$$d_{(a)}(x) = \sum_Q \zeta(Q) \delta Q(x)$$
$$N_a = \sum_Q \zeta(Q) Q, \text{ as above,}$$

and $\zeta(Q) = \pm 1$ as Q is even or odd, which under the permutations of S_n yield a modul $M(S_n; d_{(a)}(x))$. Confining our attention to standard tableaux this modul leads to the irreducible representation (a) of S_n. Specht's method of constructing the actual matrices is the same as Young's and the results are identical (cf. [21] Part IV, p. 253; for a résumé of Young's theory cf. Part III, pp. 258-269. In Part VI the theory is further developed to give the actual matrices of the representation (a) in *orthogonal form* according to a very simple rule contained in Theorems 4 and 5, pp. 217 and 218).

In [18] Specht generalizes Young's T_a to apply to any permutation group P_n. It can be shown that

$$T_a \cdot I = (f_a/n!) \sum_{P \subset S_n} \chi^{(a)}_{P^{-1}} P,$$

(cf. [21] Part IV, p. 256) where $\chi^{(a)}_{P^{-1}}$ is the characteristic of P^{-1} in (a). Specht writes

We may now write (2.1) in the following manner: [13]

$$\Delta(\alpha) = \Sigma[\Pi S_{rs}^{\lambda_{rs}}](\alpha). \tag{2.2}$$

In this form it gives the reduction of $\Delta(\alpha)$ which we are seeking.

Young [14] defines the operation S_{rs} as follows:

Y: " S_{rs} where $r < s$ *represents the operation of moving one letter from the s-th row up to the r-th row, and the resulting term is regarded as zero, whenever any row becomes less than a row below it, or when letters from the same row overlap,—as, for instance, happens when* $\alpha_1 = \alpha_2$ *in the case of* $S_{13}S_{23}$."

As an illustration we have

$$\begin{aligned}\Delta(3,2,1) &= [1 + S_{23} + S_{13} + S_{12} + S_{12}S_{23} + S_{12}S_{13} + S_{12}{}^2S_{23} + S_{12}{}^2S_{13}](3,2,1) \\ &= (3,2,1) + (3^2) + (4,2) + (4,1^2) + (4,2) + (5,1) + (5,1) + (6).\end{aligned} \tag{2.3}$$

3. The Littlewood and Richardson rule for the reduction of $\{\beta\} \times \{\gamma\}$. The second problem treated by Murnaghan [15] is the reduction of $\{\beta\} \times \{\gamma\}$ into its irreducible components $\{\alpha\}$ of order n. His method is based on Schur's expression of the characters as determinants or as quotients of alternants. This method has been used by Specht.[16]

Littlewood and Richardson have also studied this reduction. Their means

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$$X_\xi \cdot f(x) = (g_\xi/h) \sum_{P \subset P_n} \chi_{P^{-1}}^{(a)} f_{P}(x),$$

where $f(x)$ is a rational integral homogeneous function of the x_i, g_ξ is the degree of the irreducible representation of P_n, and h is the order of P_n. Corresponding to the relations amongst the T's

$$T_a \cdot T_a = T_a,$$
$$T_a \cdot T_\beta = 0,$$
$$1 = \sum_a T_a,$$

we have

$$X_\xi(X_\xi f(x)) = X_\xi f(x), \qquad X_\xi(X_\lambda f(x)) = 0, \qquad M(P_n; f(x)) = \sum_\xi M(P_n; X_\xi f(x)).$$

The function $f(x)$ is $d_{(a)}(x)$ in the case of the symmetric group, and is similarly obtainable from the tableaux in the case of the alternating and hyper-octahedral group (cf. [15] with [21] Part V),—otherwise how actually to construct it is unknown.

[13] Cf. [9].

[14] [21] Part VI, p. 199. For a changed interpretation cf. Y' at the end of § 4, which clarifies somewhat the application of Y, and is applied to the example (2.3) at the beginning of § 7.

[15] Actually he considers the corresponding problem for finite groups.

[16] [17], p. 155.

of approach is through what they call Schur or S-functions, which are none other than the characters of the $\{\alpha\}$. Their reduction of the product of two S-functions of degrees l and m into a sum of S-functions of degree n, corresponds exactly to the reduction of $\{\beta\} \times \{\gamma\}$ into its irreducible components $\{\alpha\}$. Their chief contribution to the theory is the following theorem:[17]

To every tableau which may be constructed according to the following rule there corresponds an irreducible component $\{\alpha\}$ *of* $\{\beta\} \times \{\gamma\}$, *and all such components are thereby obtained.*

LR_1: "*Take the tableau* $[\beta]$ *intact and add to it the letters of the first row of* $[\gamma]$. *These may be added to one row of* $[\beta]$, *or the symbols may be divided without disturbing their order into any number of sets, the first set being added to one row of* $[\beta]$, *the second set to a subsequent row, the third to a row subsequent to this, and so on. After the addition no row must contain more symbols than a preceding row, and no two added symbols may be in the same column.*

Next add the second row of $[\gamma]$, *according to the same rules followed by the remaining rows in succession until all the symbols of* $[\gamma]$ *have been used.*

LR_2: *These additions shall be such that each symbol of a given row of* $[\gamma]$ *in the compound tableau must appear in a later row than the letter in the same column from the preceding row of* $[\gamma]$."

In what follows we shall establish a connection with Young's equation (2. 2) which will enable us to extend the methods used by Littlewood and Richardson to give a proof of their theorem.[18]

4. A proof of the Littlewood and Richardson rule. As a first step it will be convenient to modify somewhat Young's tableau $[\alpha]$ on which the S_{rs} of (2. 2) are supposed to operate. If we interchange a pair of rows leaving the letters in the same columns as before P_α remains unaltered, and the only change induced in (2. 2) is in the interpretation of the operators S_{rs};—others amongst their products will yield the components of the right-hand side. In particular we may rearrange the rows of $[\alpha]$ so that those of $[\beta]$ come first, followed by those of $[\gamma]$, thus:

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[17] [3], p. 119.

[18] There are some slips in the application of the theorem to the reduction of $\{4, 3, 1\} \times \{2^2, 1\}$, pointed out by Murnaghan [7].

$$[\beta;\gamma]:\begin{array}{l} b_{11}\,b_{12}\cdot\cdot\cdot\, b_{1\beta_1}\\ b_{21}\cdot\cdot\, b_{2\beta_2}\\ \cdot\ \ \cdot\ \cdot\\ b_{\lambda 1}\cdot b_{\lambda\beta_\lambda}\\ c_{11}\,c_{12}\cdot\cdot\cdot\, c_{1\gamma_1}\\ c_{21}\cdot\cdot\, c_{2\gamma_2}\\ \cdot\ \ \cdot\ \cdot\\ c_{\mu 1}\cdot c_{\mu\gamma_\mu}\end{array}.$$

The corresponding representation of the full linear group L remains the same, i.e. $\{\alpha\} = \{\beta;\gamma\}$, and $(\alpha) = (\beta;\gamma)$. If we generalize the S_{rs} so that r may be greater than s, we may describe the passage from $[\alpha]$ to $[\beta;\gamma]$ by means of a product S_0 of S's. In particular $S_0 = S_{32}{}^2 S_{23}$ transforms

$$[3,2,1]:\begin{array}{lll} a & a & a\\ b & b & \\ c & & \end{array}\quad \text{into}\quad [3,1;2]:\begin{array}{lll} a & a & a\\ c & & \\ b & b & \end{array}.$$

To pass from an operator as applied to $[\alpha]$ to that as applied to $[\beta;\gamma]$ it is only necessary to multiply by $S_0{}^{-1}$ and keep track of the letters involved, which we may do by an appropriate prefix. E. g. we may write $S_0{}^{-1} = {}_bS_{23}{}^2\,{}_cS_{32}$ and the operator S_{23} as applied to $[3,2,1]$ leads to

$${}_bS_{23}{}^2\,{}_cS_{32}\cdot{}_cS_{23} = {}_bS_{23}{}^2$$

or simply $S_{23}{}^2$ as applied to $[3,1;2]$. If *after* the multiplication an operator S_{rs} remains where $r > s$, as in the case of 1 operating on $[3,2,1]$, we may *first* combine it with the other members of $S_0{}^{-1}$, ignoring the prefixes, and remultiply. The resulting tableaux will in this case not be identical with those derived from $[3,2,1]$, certain letters in the same columns being interchanged, but *the correspondence between the two sets of tablaux is unique.*[19] Corresponding to (2.3) we have

$$(4.1)\quad \Delta(3,1;2) = [S_{23} + S_{23}{}^2 + S_{23}{}^2S_{12} + S_{13} + S_{13}S_{23} + S_{23}S_{13}S_{12} + S_{13}{}^2 + S_{13}{}^2S_{12}](3,1;2).$$

In order to avoid confusion we shall write these operators S_{rs} as applied to this modification $[\beta;\gamma]$ of $[\alpha]$ as A_{rs}, the operators S_{rs} as applied to $[\beta]$ as B_{rs} and as applied to $[\gamma]$ as C_{rs}. Combining (1.4) and (2.2) we may write

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[19] A more complicated example would be 1 operating on $[4,3,2,1]$ leading to $S_{23}{}^3S_{34}{}^2S_{42} = S_{23}{}^2S_{34}$ operating on $[4,1;\ 3,2]$.

$$(4.2)\qquad \begin{aligned}\Sigma[\Pi A_{rs}^{\lambda_{rs}}]\{\beta;\gamma\} &= [\Sigma[\Pi B_{rs}^{\lambda_{rs}}]\{\beta\}] \times [\Sigma[\Pi C_{rs}^{\lambda_{rs}}]\{\gamma\}] \\ &= \{\beta\} \times [\Sigma[\Pi C_{rs}^{\lambda_{rs}}]\{\gamma\}] + \cdots \\ &\quad + \{l\} \times [\Sigma[\Pi C_{rs}^{\lambda_{rs}}]\{\gamma\}].\end{aligned}$$

From the above interpretation of S_{rs} operating on $[\beta;\gamma]$ it is clear that we may identify A_{rs} for $s \leq \lambda$ with B_{rs}; e. g. in (4.1) $S_{12} = A_{12} = B_{12}$.

Young's original restrictions Y on the S_{rs} as applied to $[\alpha]$ still hold since they are not affected by the above correspondence. Thus we may think of the tableaux arising on the left of (4.2) under the operations A_{rs} as falling into sets representative of the irreducible components [20] of

$$\{\bar{\beta}\} \times \Delta_{(\gamma)}(L),$$

and *built on* $[\bar{\beta}]$, derivable from $[\beta]$ by the B_{rs} operating according to Y. The rule of operation of the $A_{i,\lambda+j}$ we write as:

Y_1: *Take the tableaux* $[\beta]$ *intact and add to it the letters of the first row of* $[\gamma]$ *under* $A_{i,\lambda+1}$. *These may be added to one row of* $[\beta]$, *or the symbols may be divided (without disturbing their order) into any number of sets, the first set being added to one row of* $[\beta]$, *the second to a subsequent row, the third to a row subsequent to this, and so on. After the addition no row must contain more symbols than a preceding row, and no two added symbols may be in the same column.*

Next add the second row of $[\gamma]$, *according to the same rules followed by the remaining rows in succession, until all the symbols of* $[\gamma]$ *have been used.*

Y_2: *To obtain tableau built on* $[\bar{\beta}]$ *replace* $[\beta]$ *by* $[\bar{\beta}]$ *in* Y_1.

The parenthesis in Y_1 is unnecessary at this stage, in fact all the letters in a given row of $[\gamma]$ may be taken to be the same (cf. the example at the beginning of § 7), but it will be needed shortly to add definiteness to the resulting tableaux. *It is important to remark that* LR_1 *and* Y_1 *are identical.*

If we let $\beta_1 = \alpha_1$ and $\gamma_1 = \alpha_2, \cdots, \gamma_{\nu-1} = \alpha_\nu$, Y_2 becomes unnecessary and Y_1 may be written Y', which is equivalent to Y in view of the equation

$$(4.3)\qquad \Delta_{(\alpha)}(L) = \{\alpha_1\} \times \Delta_{(\alpha_2\alpha_3\ldots\alpha_\nu)}(L).$$

This change of viewpoint seems to clarify the application of Y and makes it a very simple matter to write down the tableaux representative of the irreducible components of $\Delta(\alpha)$.

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[20] By suppressing the c's we arrive at a tableau representative of $\{\bar{\beta}\}$ and we may think of $[\bar{\beta}]$ as occupying the upper left hand corner of the compound tableau, leading to the idea that this is *built on* $[\bar{\beta}]$.

5. Lattice permutations. Permutations of the m letters

$$c_1^{\gamma_1} c_2^{\gamma_2} \cdots c_\mu^{\gamma_\mu} \tag{5.1}$$

have been studied at some length, and a particular class called *lattice permutations* have been given prominence by MacMahon.[21] The definition of a lattice permutation is that amongst the first r terms of it the number of c_1's $\geq$ the number of c_2's $\geq \cdots \geq$ the number of c_μ's for all r. If we add a second suffix to the c_i's according to the order of their appearance, for each i, we may define a set of numbers which may be called *indices* of the permutation. Considering first only the c_1's and the c_2's, if c_{2s} follow c_{1t} and precede $c_{1,t+1}$ its index is defined as $s - t$ and we write

$$i_{12s} = s - t,$$

which may be positive, zero, or negative. As pointed out by Littlewood and Richardson [22] the resulting permutation of $c_1^{\gamma_1} c_2^{\gamma_2}$ is a lattice permutation if and only if no $i_{12s} > 0$. Similarly we may define indices i_{23s}, i_{34s}, etc. and *any permutation of the letters* (5.1) *is lattice if and only if no* $i_{x,x+1,s} > 0$ $(x = 1, 2, \cdots, \mu - 1)$. An important property of a lattice permutation is that by comparing it with the natural arrangement, or identical permutation, of the symbols we may uniquely associate it with a standard tableau, and conversely. E. g. with the permutation

$$1231 \text{ we associate the tableau } \begin{array}{l} 14 \\ 2 \\ 3 \end{array},$$

the lattice permutation indicating in which row the corresponding symbol is to be placed. Thus the number of distinct lattice permutations [23] of the letters (5.1) is just f_γ.

We now show how any non-lattice permutation may be associated with a lattice permutation. The steps in the process are as follows:

(a) Considering only the c_1's and the c_2's in the permutation, take the first c_2 with the greatest positive i_{12s} and change it into a c_1. Reallocating the second suffixes repeat the process, continuing until the c_1's and the c_2's are all lattice.

(b) Considering only the c_2's and the c_3's in the permutation so modified, take the first c_3 with greatest positive i_{23s} and change it into a c_2.

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[21] [4], vol. I, p. 124.

[22] [3], p. 121. Dr. A. Young has drawn my attention to the fact that the index $i_{r,r+1,s}$ is almost identical with a number used by him ([21] Part VI, § 15). In his notation $\gamma_{r+1,s,r} = -i_{r,r+1,s} + 2$ and the condition that $i_{r,r+1,s} \lesseqgtr 0$ is the same as $\gamma_{r+1,s,r} \gtreqless 2$, or that his *second tableau function* $\Pi(\gamma_{r,s,t} - 1) > 0$.

[23] Any two such we may speak of as belonging to the same *class*.

If this change upsets the 1–2 lattice property correct for it by changing a c_2 into a c_1 according to (a); *this may or may not be the new* c_2. Re-allocating the second suffixes repeat the process, continuing until the c_1's, c_2's and c_3's are all lattice.

(c) Making use of the indices $i_{34s}, i_{45s}, \cdots, i_{\mu-1,\mu s}$ proceed as above, continuing until all the c_1's, c_2's, $\cdots$, c_μ's are lattice.

This [24] we shall refer to as the *association I.*

Let us think of these changes in the light of Y' as applied to $[\gamma]$, and associate them with the operators C_{ij} in the following manner.

(a′) Changing a c_2 into a c_1 we associate with the operator C_{12}.

(b′) If changing a c_3 into a c_2 *does not* spoil the 1–2 lattice property we associate it with the operator C_{23}.

(b″) If changing a c_3 into a c_2 *does* spoil the 1–2 lattice property and we must change a c_2 into a c_1, we associate it with the operator C_{13}.

(c′) Similarly changing a c_4 is associated with C_{34}, C_{24}, or C_{14} as further changes are necessary; etc. Finally changing a c_μ is associated with $C_{\mu-1,\mu}$, $C_{\mu-2,\mu}$, $\cdots$ or $C_{1\mu}$.

Thus with each non-lattice permutation of the letters (5.1) we may also associate an operator

$$C_{12}^{\lambda_{12}} C_{13}^{\lambda_{13}} \cdots C_{\mu-1,\mu}^{\lambda_{\mu-1,\mu}} = \Pi C_{rs}^{\lambda_{rs}}, \tag{5.2}$$

which is one of those [25] applied to $[\gamma]$ under Y'. This we shall describe as the *association II.*

We are now ready to pass on to the conclusion of the proof of the Littlewood and Richardson theorem, but before doing so it will be worth while to consider in greater detail these two associations I and II in the case $\gamma_1 = \gamma_2 = \cdots = \gamma_m = 1$. The tableau $[\gamma]$ is in this case

$$\begin{matrix} 1 \\ 2 \\ 3 \\ \cdot \\ \cdot \\ m \end{matrix},$$

and each of the operators (5.2), which we shall denote $\boldsymbol{L}_2$, leads to a standard

[24] I am indebted for this association I to Mr. D. E. Littlewood.

[25] Clearly the lattice condition assures at each stage that the number of letters in any row is not less than the number in a succeeding row of the corresponding tableau. Changing the "*first* c_{r+1} *with greatest positive* $i_{r,r+1,s}$" precludes the possibility of two letters from the same row appearing in the same column, as will be clear from the following example. The permutation $c_3c_3c_1c_2c_1c_2$ leads to $c_1c_1c_1c_2c_1c_2$ under the association

tableau and a corresponding lattice permutation L_2. Thus with each of the $m!$ permutations of the letters we can associate under I a lattice permutation L_1, and under II a lattice permutation L_2, and L_1 and L_2 belong to the same class. Conversely, by reversing the steps (a), (b) and (c) we may pass backward to the permutation which [26] we may denote (S). This passage may be indicated thus:

$$1\,2\,3 \cdots m \longrightarrow L_1 \xrightarrow[(\mathbf{L}_2)^{-1}]{} (S).$$

If we interchange the rôles of the two lattice permutations L_1 and L_2 it is not difficult to see that

$$1\,2\,3 \cdots m \longrightarrow L_2 \xrightarrow[(\mathbf{L}_1)^{-1}]{} (S^{-1}).$$

Clearly if $L_1 = L_2$ then $S^2 = 1$. This remarkable duality enables us to construct a square table having Σf_γ rows and columns which is symmetrical about its leading diagonal, down which appear the Σf_γ solutions [27] of $S^2 = 1$. The remaining substitutions of S_m appear in blocks of $f_\gamma{}^2$, and $\Sigma f_\gamma{}^2 = m!$. There follows this table constructed for $m = 4$.

	1234	1231	1213	1123	1122	1212	1112	1121	1211	1111
	1	S_{14}	$S_{13}S_{34}$	$S_{12}S_{23}S_{34}$	$S_{12}S_{23}S_{24}$	$S_{13}S_{24}$	$S_{12}S_{13}S_{24}$	$S_{12}S_{23}S_{14}$	$S_{13}S_{14}$	$S_{12}S_{13}S_{14}$
1234	1234									
1231		1243 (34)	1342 (234)	2341 (1234)						
1213		1423 (243)	1324 (23)	2314 (123)						
1123		4123 (1432)	3124 (132)	2134 (12)						
1122					2143 (12) (34)	3142 (1342)				
1212					2413 (1243)	3412 (13) (24)				
1112							3214 (13)	4213 (143)	4312 (1423)	
1121							3241 (134)	4231 (14)	4132 (142)	
1211							3421 (1324)	2431 (124)	1432 (24)	
1111										4321 (14) (23)

I and to the operator $C_{12}{}^2$ under II, *not* to $c_1c_2c_1c_2c_1c_2$ and the operator $C_{12}C_{23}$ (*cf.* the end of the third paragraph on p. 122 of [3]).

[26] Assuming that the identical permutation is transformed by S into the given permutation.

[27] [2], p. 197, since all the irreducible representations of S_m are real.

6. Conclusion of the proof. We have now reached the final stage of our argument and may confine our attention to those tableaux built on $[\beta]$ which are representative of the irreducible components of

$$(6.1)\qquad \{\beta\} \times [\Sigma[\Pi C_{rs}^{\lambda_{rs}}]\{\gamma\}] = \{\beta\} \times \{\gamma\} \cdots + \{\beta\} \times \{m\}.$$

We follow Littlewood and Richardson [28] and from any such tableau read the c_{ij}'s from the right omitting the second suffix, beginning at the first row and taking the remaining rows in succession. Written in this order we have a permutation of the letters (5.1). A little consideration will show that *if a tableau built on* $[\beta]$ *according to* Y_1 *(or* LR_1*) is to satisfy* LR_2 *it is necessary and sufficient that the permutation of the c's obtained as above described should be a lattice permutation.*

We assume the Theorem to be true for all products $\{\beta\} \times \{\bar{\gamma}\}$, where $[\bar{\gamma}]$ is derivable from $[\gamma]$ under the C_{rs}, and apply an induction to prove it for $\{\beta\} \times \{\gamma\}$. That is, we assume that all tableaux satisfying the appropriate LR_2 yield the irreducible components of $\{\beta\} \times \{\bar{\gamma}\}$, since as we have seen LR_1 is automatically satisfied; this is equivalent to saying that the corresponding permutation is a lattice permutation. But clearly this is necessarily so in the case of $\{\beta\} \times \{m\}$, where all the letters of $[m]$ belong to the same row. Each non-lattice permutation of $c_1^{\gamma_1} c_2^{\gamma_2} \cdots c_\mu^{\gamma_\mu}$ is associated with an operator $\Pi C_{rs}^{\lambda_{rs}}$ under the association II, and conversely with each such operator is associated a set of tableaux built on $[\beta]$ according to LR_1 and LR_2. Thus those which remain, namely the lattice permutations, represent tableaux built on $[\beta]$ according to LR_2, and yield the irreducible components of $\{\beta\} \times \{\gamma\}$.

7. Examples of the application of the rules *Y, Y′, LR*. We may obtain the irreducible components appearing in

$$\begin{aligned}(2.3)\quad \Delta(3,2,1) &= [1+S_{23}+S_{13}+S_{12}+S_{12}S_{23}+S_{12}S_{13}+S_{12}{}^2S_{23}+S_{12}{}^2S_{13}](3,2,1)\\ &= (3,2,1)+(3^2)+(4,2)+(4,1^2)+(4,2)+(5,1)+(5,1)+(6)\end{aligned}$$

by the more systematic rule Y'. Taking the tableau $[3, 2, 1]$ to be

$$\begin{array}{lll} a & a & a \\ b & b & \\ c & & \end{array}$$

we write down the first row intact, and add the letters of the second row according to Y_1. To the resulting tableaux, namely

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[28] [3], p. 121.

$$\begin{array}{l} a\ a\ a\ b\ b, \end{array} \qquad \begin{array}{l} a\ a\ a\ b, \\ b \end{array} \qquad \begin{array}{l} a\ a\ a, \\ b\ b \end{array}$$

we must add the letter c, obtaining from the first

$$\begin{array}{l} a\ a\ a\ b\ b\ c = S_{12}{}^2 S_{13}[3, 2, 1], \end{array} \qquad \begin{array}{l} a\ a\ a\ b\ b = S_{12}{}^2 S_{23}[3, 2, 1]; \\ c \end{array}$$

from the second,

$$\begin{array}{l} a\,a\,a\,b\ c = S_{12}S_{13}[3, 2, 1], \\ b \end{array} \qquad \begin{array}{l} a\,a\,a\,b = S_{12}S_{23}[3, 2, 1], \\ b\ c \end{array} \qquad \begin{array}{l} a\,a\,a\,b = S_{12}[3, 2, 1]; \\ b \\ c \end{array}$$

and from the third,

$$\begin{array}{l} a\ a\ a\ c = S_{13}[3, 2, 1], \\ b\ b \end{array} \qquad \begin{array}{l} a\ a\ a = S_{23}[3, 2, 1], \\ b\ b\ c \end{array} \qquad \begin{array}{l} a\ a\ a = [3, 2, 1]. \\ b\ b \\ c \end{array}$$

These tableaux yield the required components.

As an illustration of LR we shall write down the tableaux representative of the irreducible components of $\{4, 2^2, 1\} \times \{2^3\}$. It will be easier to build on $[4, 2^2, 1]$, and since this tableau remains unaltered we shall represent its elements by $\bullet$'s. We write

$$[4, 2^2, 1] : \begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet, \\ \bullet\ \bullet \\ \bullet\ \bullet \\ \bullet \end{array} \qquad\qquad [2^3] : \begin{array}{l} a_1\,a_2, \\ b_1\,b_2 \\ c_1\,c_2 \end{array}$$

and begin by adding the letters of the first row of $[2^3]$ to $[4, 2^2, 1]$ according to LR_1 $(= Y_1)$. Then to these tableaux we similarly add the letters of the second and third rows of $[2^3]$, *subject at each stage to* LR_2.

$$\begin{array}{l} \bullet\,\bullet\,\bullet\,\bullet\ a_1\,a_2, \\ \bullet\,\bullet\,b_1\,b_2 \\ \bullet\,\bullet\,c_1\,c_2 \\ \bullet \end{array} \qquad \begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ a_1, a_2, \\ \bullet\ \bullet\ b_1\,b_2 \\ \bullet\ \bullet\ c_1 \\ \bullet\ c_2 \end{array} \qquad \begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ a_1\,a_2, \\ \bullet\ \bullet\,b_1\,b_2 \\ \bullet\ \bullet\,c_1 \\ \bullet \\ c_2 \end{array} \qquad \begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ a_1\,a_2; \\ \bullet\ \bullet\ b_1\,b_2 \\ \bullet\ \bullet \\ \bullet\ c_1 \\ c_2 \end{array}$$

$$\begin{array}{lllll}
\bullet\ \bullet\ \bullet\ \bullet\ a_1\,a_2, & \bullet\ \bullet\ \bullet\ \bullet\ a_1\,a_2; & \bullet\ \bullet\ \bullet\ \bullet\ a_1\,a_2, & \bullet\ \bullet\ \bullet\ \bullet\ a_1\,a_2; & \bullet\ \bullet\ \bullet\ \bullet\ a_1\,a_2: \\
\bullet\ \bullet\ b_1 & \bullet\ \bullet\ b_1 & \bullet\ \bullet\ b_1 & \bullet\ \bullet\ b_1 & \bullet\ \bullet \\
\bullet\ \bullet\ c_1 & \bullet\ \bullet & \bullet\ \bullet\ c_1 & \bullet\ \bullet & \bullet\ \bullet \\
\bullet\ b_2 & \bullet\ b_2 & \bullet & \bullet\ c_1 & \bullet\ b_1 \\
c_2 & c_1\,c_2 & b_2 & b_2 & b_2\,c_1 \\
 & & c_2 & c_2 & c_2
\end{array}$$

$$\begin{array}{llllll}
\bullet\ \bullet\ \bullet\ \bullet\ a_1, & \bullet\ \bullet\ \bullet\ \bullet\ a_1, & \bullet\ \bullet\ \bullet\ \bullet\ a_1, & \bullet\ \bullet\ \bullet\ \bullet\ a_1; & \bullet\ \bullet\ \bullet\ \bullet\ a_1, & \bullet\ \bullet\ \bullet\ \bullet\ a_1; \\
\bullet\ \bullet\ a_2\,b_1 & \bullet\ \bullet\ a_2\,b_1 & \bullet\ \bullet\ a_2\,b_1 & \bullet\ \bullet\ a_2\,b_1 & \bullet\ \bullet\ a_2\,b_1 & \bullet\ \bullet\ a_2\,b_1 \\
\bullet\ \bullet\ b_2\,c_1 & \bullet\ \bullet\ b_2\,c_1 & \bullet\ \bullet\ b_2 & \bullet\ \bullet\ b_2 & \bullet\ \bullet\ c_1 & \bullet\ \bullet \\
\bullet\ c_2 & \bullet & \bullet\ c_1\,c_2 & \bullet\ c_1 & \bullet\ b_2 & \bullet\ b_2 \\
 & c_2 & & c_2 & c_2 & c_1\,c_2
\end{array}$$

$$\begin{array}{llllll}
\bullet\ \bullet\ \bullet\ \bullet\ a_1, & \bullet\ \bullet\ \bullet\ \bullet\ a_1; & \bullet\ \bullet\ \bullet\ \bullet\ a_1, & \bullet\ \bullet\ \bullet\ \bullet\ a_1; & \bullet\ \bullet\ \bullet\ \bullet\ a_1; & \bullet\ \bullet\ \bullet\ \bullet\ a_1 \\
\bullet\ \bullet\ a_2\,b_1 & \bullet\ \bullet\ a_2\,b_1 & \bullet\ \bullet\ a_2 & \bullet\ \bullet\ a_2 & \bullet\ \bullet\ a_2 & \bullet\ \bullet\ a_2 \\
\bullet\ \bullet\ c_1 & \bullet\ \bullet & \bullet\ \bullet\ b_1 & \bullet\ \bullet\ b_1 & \bullet\ \bullet\ b_1 & \bullet\ \bullet \\
\bullet & \bullet\ c_1 & \bullet\ b_2\,c_1 & \bullet\ b_2 & \bullet\ c_1 & \bullet\ b_1 \\
b_2 & b_2 & c_2 & c_1\,c_2 & b_2 & b_2\,c_1 \\
c_2 & c_2 & & & c_2 & c_2
\end{array}$$

$$\begin{array}{lllll}
\bullet\ \bullet\ \bullet\ \bullet\ a_1, & \bullet\ \bullet\ \bullet\ \bullet\ a_1; & \bullet\ \bullet\ \bullet\ \bullet\ a_1: & \bullet\ \bullet\ \bullet\ \bullet\ a_1, & \bullet\ \bullet\ \bullet\ \bullet\ a_1; \\
\bullet\ \bullet\ b_1 & \bullet\ \bullet\ b_1 & \bullet\ \bullet & \bullet\ \bullet\ b_1 & \bullet\ \bullet\ b_1 \\
\bullet\ \bullet\ c_1 & \bullet\ \bullet & \bullet\ \bullet & \bullet\ \bullet\ c_1 & \bullet\ \bullet \\
\bullet\ a_2 & \bullet\ a_2 & \bullet\ a_2 & \bullet & \bullet\ c_1 \\
b_2 & b_2\,c_1 & b_1\,b_2 & a_2 & a_2 \\
c_2 & c_2 & c_1\,c_2 & b_2 & b_2 \\
 & & & c_2 & c_2
\end{array}$$

$$\begin{array}{lllll}
\bullet\ \bullet\ \bullet\ \bullet\ a_1: & \bullet\ \bullet\ \bullet\ \bullet, & \bullet\ \bullet\ \bullet\ \bullet; & \bullet\ \bullet\ \bullet\ \bullet, & \bullet\ \bullet\ \bullet\ \bullet; \\
\bullet\ \bullet & \bullet\ \bullet\ a_1\,a_2 & \bullet\ \bullet\ a_1\,a_2 & \bullet\ \bullet\ a_1\,a_2 & \bullet\ \bullet\ a_1\,a_2 \\
\bullet\ \bullet & \bullet\ \bullet\ b_1\,b_2 & \bullet\ \bullet\ b_1\,b_2 & \bullet\ \bullet\ b_1 & \bullet\ \bullet\ b_1 \\
\bullet\ b_1 & \bullet\ c_1\,c_2 & \bullet\ c_1 & \bullet\ b_2\,c_1 & \bullet\ b_2 \\
a_2\,c_1 & & c_2 & c_2 & c_1\,c_2 \\
b_2 & & & & \\
c_2 & & & &
\end{array}$$

$$\begin{array}{lllll}
\bullet\ \bullet\ \bullet\ \bullet; & \bullet\ \bullet\ \bullet\ \bullet: & \bullet\ \bullet\ \bullet\ \bullet, & \bullet\ \bullet\ \bullet\ \bullet; & \bullet\ \bullet\ \bullet\ \bullet: \\
\bullet\ \bullet\ a_1\,a_2 & \bullet\ \bullet\ a_1\,a_2 & \bullet\ \bullet\ a_1 & \bullet\ \bullet\ a_1 & \bullet\ \bullet\ a_1 \\
\bullet\ \bullet\ b_1 & \bullet\ \bullet & \bullet\ \bullet\ b_1 & \bullet\ \bullet\ b_1 & \bullet\ \bullet \\
\bullet\ c_1 & \bullet\ b_1 & \bullet\ a_2\,c_1 & \bullet\ a_2 & \bullet\ a_2 \\
b_2 & b_2\,c_1 & b_2 & b_2\,c_1 & b_1\,b_2 \\
c_2 & c_2 & c_2 & c_2 & c_1\,c_2
\end{array}$$

$$
\begin{array}{llll}
\bullet & \bullet & \bullet & \bullet\,; \\
\bullet & \bullet & a_1 & \\
\bullet & \bullet & b_1 & \\
\bullet & c_1 & & \\
a_2 & & & \\
b_2 & & & \\
c_2 & & &
\end{array}
\qquad\qquad
\begin{array}{llll}
\bullet & \bullet & \bullet & \bullet\,: \\
\bullet & \bullet & a_1 & \\
\bullet & \bullet & & \\
\bullet & b_1 & & \\
a_2 & c_1 & & \\
b_2 & & & \\
c_2 & & &
\end{array}
\qquad\qquad
\begin{array}{llll}
\bullet & \bullet & \bullet & \bullet\,. \\
\bullet & \bullet & & \\
\bullet & \bullet & & \\
\bullet & a_1 & & \\
a_2 & b_1 & & \\
b_2 & c_1 & & \\
c_2. & & &
\end{array}
$$

While the number of tableaux which it is necessary to construct in a given product may not be small, nevertheless after a little practice the application of LR becomes quite mechanical, and is entirely elementary. Each tableau representing an irreducible component, we have the equation

$$
\begin{aligned}
\{4, 2^2, 1\} \times \{2^3\} = {} & (6, 4^2, 1) + (6, 4, 3, 2) + (6, 4, 3, 1^2) + (6, 4, 2^2, 1) \\
& + (6, 3^2, 2, 1) + (6, 3, 2^3) + (6, 3^2, 1^3) + (6, 3, 2^2, 1^2) \\
& + (6, 2^4, 1) + (5, 4^2, 2) + (5, 4^2, 1^2) + (5, 4, 3^2) \\
& + 2(5, 4, 3, 2, 1) + (5, 4, 2^3) + (5, 4, 3, 1^3) \\
& + (5, 4, 2^2, 1^2) + (5, 3^3, 1) + (5, 3^2, 2^2) + (5, 3^2, 2, 1^2) \\
& + (5, 3, 2^3, 1) + (5, 3^2, 2, 1^2) + (5, 3, 2^3, 1) + (5, 2^5) \\
& + (5, 3^2, 1^4) + (5, 3, 2^2, 1^3) + (5, 2^4, 1^2) + (4^3, 3) \\
& + (4^3, 2, 1) + (4^2, 3^2, 1) + (4^2, 3, 2^2) + (4^2, 3, 2, 1^2) \\
& + (4^2, 2^3, 1) + (4, 3^3, 1^2) + (4, 3^2, 2^2, 1) + (4, 3, 2^4) \\
& + (4, 3^2, 2, 1^3) + (4, 3, 2^3, 1^2) + (4, 2^5, 1).
\end{aligned}
$$

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THE ANALYSIS OF THE KRONECKER PRODUCT OF IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP.*

By F. D. Murnaghan.

Introduction. The well known theorems which are known as the Clebsch-Gordan series, and which are fundamental in invariant theory and in the application of group theory to quantum mechanics, are two in number. The first furnishes the analysis of the Kronecker product of any two irreducible representations of the two-dimensional unimodular linear group into its irreducible components; and the second furnishes the analysis of the Kronecker product of any two irreducible representations of the three-dimensional rotation group. The irreducible representations (rational-integral) of the n-dimensional linear group are each described by $j \leq n$, positive integers (whose sum is the degree of the representation in question); hence when $n = 2$ j has only two possible values 1 and 2 (the identity representation, of degree zero, is exceptional, being characterised by the number zero). A further simplification is introduced by the demand that the two-dimensional linear group be *unimodular:* in this case each irreducible representation (continuous) is rational-integral and is characterised by a single label (a positive integer or zero). Thus the first of the two classical Clebsch-Gordan series furnishes the analysis of the Kronecker product $(m) \times (n)$ of two irreducible representations (m) and (n) of the two-dimensional unimodular linear group, each term in the analysis being described by a single non-negative integer. The irreducible representations of the n-dimensional rotation group are each described by $j \leq k$ positive integers where $k = n/2$ if n is even and $k = (n-1)/2$ when n is odd (the identity representation, which is characterised by the number zero, being again exceptional). Hence when $n = 3$ each irreducible representation is again characterised by a single label and the second of the two classical Clebsch-Gordan series furnishes the analysis of the Kronecker product $(m) \times (n)$ of two irreducible representations (m) and (n) of the three-dimensional rotation group, each term in the analysis being described by a single non-negative integer. In a recent paper (1) Brauer describes a method for finding the analysis of the Kronecker product of two irreducible representations of a semi-simple continuous group and makes the following remark: "Dans le cas d'un groupe d'ordre fini, on ne connait pas de loi explicite de

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cette nature." We give in this paper the analysis of the Kronecker product of various pairs of irreducible representations of the symmetric group on n letters. Each irreducible representation of this group is described by $j \leq n$ positive integers, whose sum is n, so that a restriction analogous to that imposed in the classical Clebsch-Gordan series would be to demand that $n = 2$ which would reduce the question to a triviality. Fortunately no such restriction is necessary: the coefficients in the analysis are *independent of n*. This is the central result of the present paper and we think it well to add a few words of explanation here to make what is meant quite clear. The simplest non-trivial instance of the theory furnishes the analysis of $(n-1,1) \times (n-1,1)$ i. e. the Kronecker square of the irreducible representation $(n-1,1)$ of the symmetric group on n letters (the representation (n), described by the *single* positive integer n, being the identity representation, acts as a unit in the formation of Kronecker products). The result, *valid for arbitrary n*, is

$$(n-1,1) \times (n-1,1) = (n) + (n-1,1) + (n-2,2) + (n-2,1^2).$$

However, one must understand that if a partition of n which occurs in the analysis is disordered, i. e. not in normal non-increasing order, it must be restored to this normal order, or discarded, in the manner described in detail in our previous paper (**2**) p. 461. Thus when $n = 2$ we obtain on the right the two disordered terms $(0,2)$ and $(0,1^2)$; of these the first must be replaced by $-(1^2)$ and the second discarded (because it contains an element, 1, which is greater than the preceding element, 0, by unity). Hence $(1^2) \times (1^2) = (2) + (1^2) - (1^2) = (2)$ which is the trivially evident fact that the Kronecker square of the alternating representation of the symmetric group on two letters is the identity representation of this group. When $n = 3$ we obtain amongst the four terms in the analysis the term $(1,2)$ which must be discarded so that

$$(2,1) \times (2,1) = (3) + (2,1) + (1^3).$$

For higher values of n no disordered partitions of n occur in the analysis. Thus

$$n = 4;\ (3,1) \times (3,1) = (4) + (3,1) + (2^2) + (2,1^2)$$
$$n = 5;\ (4,1) \times (4,1) = (5) + (4,1) + (3,2) + (3,1^2)$$

and so on. We give in the present paper the analyses of $(n-p, \lambda_2, \cdots) \times (n-q, \mu_2, \cdots)$ for the following values of p and q: $p = 1$, $q = 1, 2, 3, 4, 5$ (there being 18 of these); $p = 2$, $q = 2, 3, 4$ (there being 19 of these); $p = 3$, $q = 3, 4$ (there being 21 of these). It is not surprising that the formulae become increasingly complicated as $p + q$ increases; in fact, the analysis of

$(n-p, \lambda_2, \cdots) \times (n-q, \mu_2, \cdots)$ does not contain any component whose initial element $< n-p-q$. Those we give suffice (in combination with certain elementary and obvious devices) to give the analysis of all products provided $n \leq 8$ and we append these analyses. It would have been possible to construct these analyses from the character tables (by a Fourier analysis method) but such a method is exceedingly tedious and anyone attempting it inevitably feels himself classed amongst the "hewers of wood and drawers of water." By availing ourselves of the two central facts

(a) that the coefficients of the analysis are independent of n and
(b) that the analysis of $(n-p, \cdots) \times (n-q, \cdots)$ does not go deeper than the terms $(n-p-q, \cdots)$

we are able to dispense with the troublesome Fourier analysis. The page references in the following paragraph refer to our paper (**2**). We close this introduction with the doubtless unnecessary remark that the Kronecker products discussed here must not be confused with the "direct" products treated in a previous paper (**3**). The direct product of two irreducible representations of the symmetric groups on m and n letters, respectively, may be regarded as a Kronecker product but *not* of representations of the symmetric groups; we have to take the *attached* irreducible representations, of degrees m and n, respectively, of the full linear group and then the Kronecker product, of degree $m+n$, of these irreducible representations of the full linear group (*not* the symmetric group) is *attached* to the direct product of the two given representations of the symmetric group.

1. Description of the method. The essential fact that the coefficients of the analysis of $(n-p, \lambda_2, \cdots) \times (n-q, \mu_2, \cdots)$ depend only on $(\lambda_2, \cdots)$, $(\mu_2, \cdots)$, whose sums are p and q, respectively, and *not* on n is an immediate consequence of a remark made in a previous paper (**4**) in which we gave explicit formulae for the characters of various representations $(n-p, \lambda_2, \cdots)$ of the symmetric group on n letters in terms of the class numbers $(\alpha) = (\alpha_1, \alpha_2, \cdots)$. We pointed out there that the formulae of this type did not involve n explicitly; they are in fact polynomials in $(\alpha_1, \alpha_2, \cdots)$ with coefficients depending on $(\lambda_2, \cdots)$. Since the characters of a Kronecker product are simply the products of the characters of the factors the fact that the coefficients occurring in the analysis of our Kronecker product are independent of n is a foregone conclusion. The rule for rearranging disordered partitions of n is an immediate consequence of the following fact (p. 460): if $(\lambda) = (\lambda_1, \cdots)$ is any partition of n the characteristic $\phi_{(\lambda)}(s)$ of the irreducible representation (λ) is a determinant with the property

$$\phi_{(\ldots\lambda_p,\lambda_{p+1},\ldots)}(s) = -\phi_{(\ldots\lambda_{p+1}-1,\lambda_p+1\ldots)}(s).$$

Since

$$\phi_{(\lambda)}(s) = \frac{1}{n!}\Sigma\, n_{(a)}\chi_{(\lambda)}^{(a)}s^{(a)}$$

it follows that

$$\chi^{(a)}_{(\ldots\lambda_p,\lambda_{p+1},\ldots)} = -\chi^{(a)}_{(\ldots\lambda_{p+1}-1,\lambda_p+1,\ldots)}$$

(Cf. end of § 1, p. 742, of our previous paper (**4**)).

It remains to show that the product $(n-p,\lambda_2,\cdots)\times(n-q,\mu_2,\cdots)$ does not contain any terms whose first element is less than $n-p-q$. To do this we observe that the characters of $\Delta(n)$, the identity representation, are 1; that those of $\Delta(n-1,1)$, whose characteristic is $q_{n-1}(s)s_1$, are α_1 (see (**4**) p. 740); that those of $\Delta(n-2,1^2)$, whose characteristic is $q_{n-2}(s)s_1^2$, are $\alpha_1(\alpha_1-1)$ and so on. Since any polynomial in α_1 is a linear combination of the quantities $1, \alpha_1, \alpha_1(\alpha_1-1),\cdots$, it follows that any polynomial in α_1 of degree m is a linear combination of the characters of the representations $\Delta(n-j,1^j)$, $j=0,1,\cdots,m$. Again the representation $\Delta(n-2,2)$, whose characteristic is $q_{n-2}(s)q_2(s)=\frac{1}{2}q_{n-2}(s)(s_1^2+s_2)$ has $\frac{1}{2}\alpha_1(\alpha_1-1)+\alpha_2$ for its characters so that α_2 is a linear combination of the characters of representations which go no deeper than $\Delta(n-2,\cdots)$. On considering $\Delta(n-2-j,2,1^j)$ whose characteristic is

$$q_{n-2-j}(s)q_2(s)q_1{}^j(s) = \tfrac{1}{2}q_{n-2-j}(s)(s_1^{j+2}+s_1{}^js_2)$$

we see that the product of any polynomial in α_1 of degree m by α_2 is a linear combination of the characters of representations $\Delta(\lambda)$, the representations involved going no deeper than $\Delta(n-m-2,\cdots)$. It would be pedantic to continue the argument; it being now clear that any polynomial in $(\alpha_1,\cdots,\alpha_n)$ whose term of greatest weight is $k\leq n$ (the weight w of any term $\alpha_1^{p_1}\cdots\alpha_n^{p_n}$ being defined by $p_1+2p_2+\cdots+np_n=w$) is a linear combination of the characters of representations $\Delta(\lambda)$, the representations involved going no deeper than $\Delta(n-k,\cdots)$. But the expression for the characters of $D(n-p,\cdots)$ has no term of weight $>p$; hence the expression for the characters of the product $D(n-p,\cdots)\times D(n-q,\cdots)$ has no term of weight $>p+q$. In other words the compound representation $D(n-p,\cdots)\times D(n-q,\cdots)$ is a linear combination (in fact with integral, but not necessarily positive, coefficients) of the (compound) representations $\Delta(\lambda)$, no representations deeper than $\Delta(n-p-q,\cdots)$ being involved. This implies (cf. (**2**) pp. 469-479) that the product $D(n-p,\cdots)\times D(n-q,\cdots)$ is a linear combination (with positive integral coefficients) of irreducible representations $D(\lambda)$, no representations deeper than $D(n-p-q,\cdots)$ being involved.

We shall illustrate in detail the method of determining the coefficients occurring in the desired analysis by considering the first two instances. We first observe that if we write $(\lambda) \times (\mu) = g^{(\alpha)}_{(\lambda)(\mu)}(\alpha)$ (summation convention!) the coefficient symbol $g^{(\alpha)}_{(\lambda)(\mu)}$ is symmetric in its lower labels:

$$g^{(\alpha)}_{(\lambda)(\mu)} = g^{(\alpha)}_{(\mu)(\lambda)}$$

simply because Kronecker multiplication is commutative. More than this $g^{(\alpha)}_{(\lambda)(\mu)}$ is symmetric in all three labels (λ), (μ), (α). In fact the relation $(\lambda) \times (\mu) = g^{(\alpha)}_{(\lambda)(\mu)}(\alpha)$ which serves to define the integers $g^{(\alpha)}_{(\lambda)(\mu)}$ forces

$$\chi^{(j)}_{(\lambda)}\chi^{(j)}_{(\mu)} = g^{(\alpha)}_{(\lambda)(\mu)}\chi^{(j)}_{(\alpha)}$$

where (j) is any class of the symmetric group on n letters. On multiplication by $n_j\chi^{(j)}_{(\nu)} \div n!$ and summation over all classes (j) we obtain (owing to the orthogonality relations amongst the characters and the fact that they are all real)

$$g^{(\nu)}_{(\lambda)(\mu)} = \frac{1}{n!}\sum_{(j)}\chi^{(j)}_{(\lambda)}\chi^{(j)}_{(\mu)}\chi^{(j)}_{(\nu)}$$

proving the symmetry of the coefficient symbol $g^{(\nu)}_{(\lambda)(\mu)}$ in all three labels (λ), (μ), (ν). In other words the coefficient of (ν) in the analysis of $(\lambda) \times (\mu)$ is the same as the coefficient of (μ) in the analysis of $(\lambda) \times (\nu)$. If in particular (ν) is the identity representation (n) we see that (n) does not appear in any Kronecker product $(\lambda) \times (\mu)$ unless it is a Kronecker square $(\lambda) \times (\lambda)$ and in this case it occurs exactly once (for $(\lambda) \times (n) = (\lambda)$).

These remarks are sufficient to enable us to proceed with our, now trivial, calculations. Thus to analyse $(n-1,1) \times (n-1,1)$ we write it as follows:

$$(n-1,1) \times (n-1,1) = (n) + c_1(n-1,1) + c_2(n-2,2) + c_{1^2}(n-2,1^2)$$

our problem being the determination of the three coefficients c_1, c_2, c_{1^2} which are independent of n. Setting $n=0$ we find $c_1 = c_{1^2}$, since

$$(-1,1) = -(0),\ (-2,2) = -(1,-1) = 0,\ (-2,1^2) = -(0,-1,1) = (0).$$

There is of course no symmetric group when $n=0$; but $q_0(s) = 1$ and we may regard (0) as 1. Setting $n=1$ we obtain $c_2 = 1$; for $(0,1)$ vanishes as also does $(-1,1^2)$ since this last contains an element (the third) which is greater by 2 than an element (the first) two steps ahead of it. Finally setting $n=2$ we find $c_1 = c_2$; for the product of (1^2) by (1^2) is the associate, namely (2), of (1^2) and $(0,1^2)$ vanishes. Hence $c_1 = c_2 = c_{1^2} = 1$ so that the desired analysis is

$$(n-1,1)\times(n-1,1)=(n)+(n-1,1)+(n-2,2)+(n-2,1^2).$$

In proceeding with the next example $(n-1,1)\times(n-2,2)$ we know that the coefficient of (n) is zero whilst that of $(n-1,1)$ is 1. Hence we write

$$(n-1,1)\times(n-2,2)=(n-1,1)+c_2(n-2,2)+c_{1^2}(n-2,1^2)$$
$$+c_3(n-3,3)+c_{2,1}(n-3,2,1)+c_{1^3}(n-3,1^3).$$

Setting $n=0$ we get $c_{1^2}-c_{1^3}=1$, since $(-2,2)$ vanishes, as do also $(-3,3)$, $(-3,2,1)$, whilst $(-1,1)=-(0)$, $(-2,1^2)=(0)$, $(-3,1^3)=-(0)$. Setting $n=1$ we find $c_2=c_{2,1}$ since $(0,1)$ vanishes, as do also $(-1,1^2)$, $(-2,3)$, $(-2,1^3)$, whilst $(-1,2)=-(1)$ and $(-2,2,1)=(1)$. Setting $n=2$ the left hand side becomes the associate of $(0,2)=-(1^2)$ i. e. $-(2)$. The right hand side becomes $(1-c_2)(1^2)-c_3(2)$ and since the simple characteristics of the symmetric group on two letters are linearly independent we obtain the *two* relations $c_2=1$; $c_3=1$; setting $n=3$ (in which case the factor $(1,2)$ on the left vanishes) we obtain the two relations: $c_3=1$, $c_{2,1}=c_{1^3}$ so that $c_2=1$, $c_{1^2}=1$, $c_3=1$, $c_{2,1}=1$, $c_{1^3}=0$. The various analyses given in the next section were obtained in this way.

2. The results of the analysis. Denoting by $(\lambda_1,\lambda_2,\cdots,\lambda_j)$ the irreducible representation $D(\lambda)$ of the symmetric group on $n=\lambda_1+\lambda_2+\cdots+\lambda_j$ letters we have:

1). $(n-1,1)\times(n-1,1)=(n)+(n-1,1)+(n-2,2)+(n-2,1^2)$.

2). $(n-1,1)\times(n-2,2)=(n-1,1)+(n-2,2)+(n-2,1^2)$
$+(n-3,3)+(n-3,2,1)$.

3). $(n-1,1)\times(n-2,1^2)=(n-1,1)+(n-2,2)+(n-2,1^2)$
$+(n-3,2,1)+(n-3,1^3)$.

4). $(n-1,1)\times(n-3,3)=(n-2,2)+(n-3,3)+(n-3,2,1)$
$+(n-4,4)+(n-4,3,1)$.

5). $(n-1,1)\times(n-3,2,1)=(n-2,2)+(n-2,1^2)+(n-3,3)$
$+2(n-3,2,1)+(n-3,1^3)$
$+(n-4,3,1)+(n-4,2^2)$
$+(n-4,2,1^2)$.

6). $(n-1,1)\times(n-3,1^3)=(n-2,1^2)+(n-3,2,1)+(n-3,1^3)$
$+(n-4,2,1^2)+(n-4,1^4)$.

7). $(n-1,1)\times(n-4,4)=(n-3,3)+(n-4,4)+(n-4,3,1)$
$+(n-5,5)+(n-5,4,1)$.

8). $(n-1,1)\times(n-4,3,1)=(n-3,3)+(n-3,2,1)+(n-4,4)$
$+2(n-4,3,1)+(n-4,2^2)$

$$+ (n-4, 2, 1^2) + (n-5, 4, 1)$$
$$+ (n-5, 3, 2) + (n-5, 3, 1^2).$$

9). $(n-1, 1) \times (n-4, 2^2) = (n-3, 2, 1) + (n-4, 3, 1) + (n-4, 2^2)$
$$+ (n-4, 2, 1^2) + (n-5, 3, 2)$$
$$+ (n-5, 2^2, 1).$$

10). $(n-1, 1) \times (n-4, 2, 1^2) = (n-3, 2, 1) + (n-3, 1^3)$
$$+ (n-4, 3, 1) + (n-4, 2^2)$$
$$+ 2(n-4, 2, 1^2) + (n-4, 1^4)$$
$$+ (n-5, 3, 1^2) + (n-5, 2^2, 1)$$
$$+ (n-5, 2, 1^3).$$

11). $(n-1, 1) \times (n-4, 1^4) = (n-3, 1^3) + (n-4, 2, 1^2) + (n-4, 1^4)$
$$+ (n-5, 2, 1^3) + (n-5, 1^5).$$

12). $(n-1, 1) \times (n-5, 5) = (n-4, 4) + (n-5, 5) + (n-5, 4, 1)$
$$+ (n-6, 6) + (n-6, 5, 1).$$

13). $(n-1, 1) \times (n-5, 4, 1) = (n-4, 4) + (n-4, 3, 1) + (n-5, 5)$
$$+ 2(n-5, 4, 1) + (n-5, 3, 2)$$
$$+ (n-5, 3, 1^2) + (n-6, 5, 1)$$
$$+ (n-6, 4, 2) + (n-6, 4, 1^2)$$

14). $(n-1, 1) \times (n-5, 3, 2) = (n-4, 3, 1) + (n-4, 2^2) + (n-5, 4, 1)$
$$+ 2(n-5, 3, 2) + (n-5, 3, 1^2)$$
$$+ (n-5, 2^2, 1) + (n-6, 4, 2)$$
$$+ (n-6, 3^2) + (n-6, 3, 2, 1).$$

15). $(n-1, 1) \times (n-5, 3, 1^2) = (n-4, 3, 1) + (n-4, 2, 1^2)$
$$+ (n-5, 4, 1) + (n-5, 3, 2)$$
$$+ 2(n-5, 3, 1^2) + (n-5, 2^2, 1)$$
$$+ (n-5, 2, 1^3) + (n-6, 4, 1^2)$$
$$+ (n-6, 3, 2, 1) + (n-6, 3, 1^3).$$

16). $(n-1, 1) \times (n-5, 2^2, 1) = (n-4, 2^2) + (n-4, 2, 1^2) + (n-5, 3, 2)$
$$+ (n-5, 3, 1^2) + 2(n-5, 2^2, 1)$$
$$+ (n-5, 2, 1^3) + (n-6, 3, 2, 1)$$
$$+ (n-6, 2^3) + (n-6, 2^2, 1^2).$$

17). $(n-1, 1) \times (n-5, 2, 1^3) = (n-4, 2, 1^2) + (n-4, 1^4) + (n-5, 3, 1^2)$
$$+ (n-5, 2^2, 1) + 2(n-5, 2, 1^3)$$
$$+ (n-5, 1^5) + (n-6, 3, 1^3)$$
$$+ (n-6, 2^2, 1^2) + (n-6, 2, 1^4).$$

18). $(n-1, 1) \times (n-5, 1^5) = (n-4, 1^4) + (n-5, 2, 1^3) + (n-5, 1^5)$
$$+ (n-6, 2, 1^4) + (n-6, 1^6).$$

19). $(n-2,2)\times(n-2,2)=(n)+(n-1,1)+2(n-2,2)+(n-2,1^2)$
$+(n-3,3)+2(n-3,2,1)$
$+(n-3,1^3)+(n-4,4)$
$+(n-4,3,1)+(n-4,2^2).$

20). $(n-2,2)\times(n-2,1^2)=(n-1,1)+(n-2,2)+2(n-2,1^2)$
$+(n-3,3)+2(n-3,2,1)$
$+(n-3,1^3)+(n-4,3,1)$
$+(n-4,2,1^2).$

21). $(n-2,2)\times(n-3,3)=(n-1,1)+(n-2,2)+(n-2,1^2)$
$+2(n-3,3)+2(n-3,2,1)$
$+(n-4,4)+2(n-4,3,1)$
$+(n-4,2^2)+(n-4,2,1^2)$
$+(n-5,5)+(n-5,4,1)$
$+(n-5,3,2).$

22). $(n-2,2)\times(n-3,2,1)=(n-1,1)+2(n-2,2)+2(n-2,1^2)$
$+2(n-3,3)+4(n-3,2,1)$
$+2(n-3,1^3)+(n-4,4)$
$+3(n-4,3,1)+2(n-4,2^2)$
$+3(n-4,2,1^2)+(n-4,1^4)$
$+(n-5,4,1)+(n-5,3,2)$
$+(n-5,3,1^2)+(n-5,2^2,1).$

23). $(n-2,2)\times(n-3,1^3)=(n-2,2)+(n-2,1^2)+2(n-3,2,1)$
$+2(n-3,1^3)+(n-4,3,1)$
$+(n-4,2^2)+2(n-4,2,1^2)$
$+(n-4,1^4)+(n-5,3,1^2)$
$+(n-5,2,1^3).$

24). $(n-2,2)\times(n-4,4)=(n-2,2)+(n-3,3)+(n-3,2,1)$
$+2(n-4,4)+2(n-4,3,1)$
$+(n-4,2^2)+(n-5,5)$
$+2(n-5,4,1)+(n-5,3,2)$
$+(n-5,3,1^2)+(n-6,6)$
$+(n-6,5,1)+(n-6,4,2).$

25). $(n-2,2)\times(n-4,3,1)=(n-2,2)+(n-2,1^2)+2(n-3,3)$
$+3(n-3,2,1)+(n-3,1^3)$
$+2(n-4,4)+5(n-4,3,1)$
$+2(n-4,2^2)+3(n-4,2,1^2)$
$+(n-5,5)+3(n-5,4,1)$
$+3(n-5,3,2)+3(n-5,3,1^2)$
$+2(n-5,2^2,1)+(n-5,2,1^3)$

$$+ (n-6,5,1) + (n-6,4,2) + (n-6,4,1^2) + (n-6,3^2) + (n-6,3,2,1).$$

26). $(n-2,2) \times (n-4,2^2) = (n-2,2) + (n-3,3) + 2(n-3,2,1) + (n-3,1^3) + (n-4,4) + 2(n-4,3,1) + 3(n-4,2^2) + 2(n-4,2,1^2) + (n-4,1^4) + (n-5,4,1) + 2(n-5,3,2) + 2(n-5,3,1^2) + 2(n-5,2^2,1) + (n-5,2,1^3) + (n-6,4,2) + (n-6,3,2,1) + (n-6,2^3).$

27). $(n-2,2) \times (n-4,2,1^2) = (n-2,1^2) + (n-3,3) + 3(n-3,2,1) + 2(n-3,1^3) + 3(n-4,3,1) + 2(n-4,2^2) + 5(n-4,2,1^2) + 2(n-4,1^4) + (n-5,4,1) + 2(n-5,3,2) + 3(n-5,3,1^2) + 3(n-5,2^2,1) + 3(n-5,2,1^3) + (n-5,1^5) + (n-6,4,1^2) + (n-6,3,2,1) + (n-6,3,1^3) + (n-6,2^2,1^2).$

28). $(n-2,2) \times (n-4,1^4) = (n-3,2,1) + (n-3,1^3) + (n-4,2^2) + 2(n-4,2,1^2) + 2(n-4,1^4) + (n-5,3,1^2) + (n-5,2^2,1) + 2(n-5,2,1^3) + (n-5,1^5) + (n-6,3,1^3) + (n-6,2,1^4).$

29). $(n-2,1^2) \times (n-2,1^2) = (n) + (n-1,1) + 2(n-2,2) + (n-2,1^2) + (n-3,3) + 2(n-3,2,1) + (n-3,1^3) + (n-4,2^2) + (n-4,2,1^2) + (n-4,1^4).$

30). $(n-2,1^2) \times (n-3,3) = (n-2,2) + (n-2,1^2) + (n-3,3) + 2(n-3,2,1) + (n-3,1^3) + (n-4,4) + 2(n-4,3,1) + (n-4,2^2) + (n-4,2,1^2) + (n-5,4,1) + (n-5,3,1^2).$

31). $(n-2,1^2) \times (n-3,2,1) = (n-1,1) + 2(n-2,2) + 2(n-2,1^2) + 2(n-3,3) + 4(n-3,2,1) + 2(n-3,1^3) + (n-4,4) + 3(n-4,3,1) + 2(n-4,2^2)$

$$+3(n-4,2,1^2)+(n-4,1^4)$$
$$+(n-5,3,2)+(n-5,3,1^2)$$
$$+(n-5,2^2,1)+(n-5,2,1^3).$$

32). $(n-2,1^2)\times(n-3,1^3)=(n-1,1)+(n-2,2)+(n-2,1^2)$
$$+(n-3,3)+2(n-3,2,1)$$
$$+(n-3,1^3)+(n-4,3,1)$$
$$+(n-4,2^2)+2(n-4,2,1^2)$$
$$+(n-4,1^4)+(n-5,2^2,1)$$
$$+(n-5,2,1^3)+(n-5,1^5).$$

33). $(n-2,1^2)\times(n-4,4)=(n-3,3)+(n-3,2,1)+(n-4,4)$
$$+2(n-4,3,1)+(n-4,2,1^2)$$
$$+(n-5,5)+2(n-5,4,1)$$
$$+(n-5,3,2)+(n-5,3,1^2)$$
$$+(n-6,5,1)+(n-6,4,1^2).$$

34). $(n-2,1^2)\times(n-4,3,1)=(n-2,2)+2(n-3,3)+3(n-3,2,1)$
$$+(n-3,1^3)+2(n-4,4)$$
$$+4(n-4,3,1)+3(n-4,2^2)$$
$$+3(n-4,2,1^2)+(n-4,1^4)$$
$$+(n-5,5)+3(n-5,4,1)$$
$$+3(n-5,3,2)+3(n-5,3,1^2)$$
$$+2(n-5,2^2,1)+(n-5,2,1^3)$$
$$+(n-6,4,2)+(n-6,4,1^2)$$
$$+(n-6,3,2,1)+(n-6,3,1^3).$$

35). $(n-2,1^2)\times(n-4,2^2)=(n-2,1^2)+(n-3,3)+2(n-3,2,1)$
$$+(n-3,1^3)+3(n-4,3,1)$$
$$+(n-4,2^2)+3(n-4,2,1^2)$$
$$+(n-5,4,1)+2(n-5,3,2)$$
$$+2(n-5,3,1^2)+2(n-5,2^2,1)$$
$$+(n-5,2,1^3)+(n-6,3^2)$$
$$+(n-6,3,2,1)+(n-6,2^2,1^2).$$

36). $(n-2,1^2)\times(n-4,2,1^2)=(n-2,2)+(n-2,1^2)+(n-3,3)$
$$+3(n-3,2,1)+2(n-3,1^3)$$
$$+(n-4,4)+3(n-4,3,1)$$
$$+3(n-4,2^2)+4(n-4,2,1^2)$$
$$+2(n-4,1^4)+(n-5,4,1)$$
$$+2(n-5,3,2)+3(n-5,3,1^2)$$
$$+3(n-5,2^2,1)+3(n-5,2,1^3)$$
$$+(n-5,1^5)+(n-6,3,2,1)$$
$$+(n-6,3,1^3)+(n-6,2^3)$$
$$+(n-6,2^2,1^2)+(n-6,2,1^4).$$

37). $$\begin{aligned}(n-2,1^2)\times(n-4,1^4) = {} & (n-2,1^2)+(n-3,2,1)+(n-3,1^3)\\ &+(n-4,3,1)+2(n-4,2,1^2)\\ &+(n-4,1^4)+(n-5,3,1^2)\\ &+(n-5,2^2,1)+2(n-5,2,1^3)\\ &+(n-5,1^5)+(n-6,2^2,1^2)\\ &+(n-6,2,1^4)+(n-6,1^6).\end{aligned}$$

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38). $$\begin{aligned}(n-3,3)\times(n-3,3) = {} & (n)+(n-1,1)+2(n-2,2)+(n-2,1^2)\\ &+2(n-3,3)+2(n-3,2,1)\\ &+(n-3,1^3)+2(n-4,4)\\ &+3(n-4,3,1)+2(n-4,2^2)\\ &+(n-4,2,1^2)+(n-5,5)\\ &+2(n-5,4,1)+2(n-5,3,2)\\ &+(n-5,3,1^2)+(n-5,2^2,1)\\ &+(n-6,6)+(n-6,5,1)\\ &+(n-6,4,2)+(n-6,3^2).\end{aligned}$$

39). $$\begin{aligned}(n-3,3)\times(n-3,2,1) = {} & (n-1,1)+2(n-2,2)+2(n-2,1^2)\\ &+2(n-3,3)+5(n-3,2,1)\\ &+2(n-3,1^3)+2(n-4,4)\\ &+5(n-4,3,1)+3(n-4,2^2)\\ &+4(n-4,2,1^2)+(n-4,1^4)\\ &+(n-5,5)+3(n-5,4,1)\\ &+3(n-5,3,2)+3(n-5,3,1^2)\\ &+2(n-5,2^2,1)+(n-5,2,1^3)\\ &+(n-6,5,1)+(n-6,4,2)\\ &+(n-6,4,1^2)+(n-6,3,2,1).\end{aligned}$$

40). $$\begin{aligned}(n-3,3)\times(n-3,1^3) = {} & (n-2,1^2)+(n-3,3)+2(n-3,2,1)\\ &+2(n-3,1^3)+2(n-4,3,1)\\ &+(n-4,2^2)+3(n-4,2,1^2)\\ &+(n-4,1^4)+(n-5,4,1)\\ &+(n-5,3,2)+2(n-5,3,1^2)\\ &+(n-5,2^2,1)+(n-5,2,1^3)\\ &+(n-6,4,1^2)+(n-6,3,1^3).\end{aligned}$$

41). $$\begin{aligned}(n-3,3)\times(n-4,4) = {} & (n-1,1)+(n-2,2)+(n-2,1^2)\\ &+2(n-3,3)+2(n-3,2,1)\\ &+2(n-4,4)+3(n-4,3,1)\\ &+(n-4,2^2)+(n-4,2,1^2)\\ &+2(n-5,5)+3(n-5,4,1)\\ &+3(n-5,3,2)+(n-5,3,1^2)\end{aligned}$$

$$+ (n-5, 2^2, 1) + (n-6, 6) + 2(n-6, 5, 1) + 2(n-6, 4, 2) + (n-6, 4, 1^2) + (n-6, 3^2) + (n-6, 3, 2, 1) + (n-7, 7) + (n-7, 6, 1) + (n-7, 5, 2) + (n-7, 4, 3).$$

42). $(n-3, 3) \times (n-4, 3, 1) = (n-1, 1) + 2(n-2, 2) + 2(n-2, 1^2) + 3(n-3, 3) + 5(n-3, 2, 1) + 2(n-3, 1^3) + 3(n-4, 4) + 7(n-4, 3, 1) + 4(n-4, 2^2) + 5(n-4, 2, 1^2) + (n-4, 1^4) + 2(n-5, 5) + 6(n-5, 4, 1) + 6(n-5, 3, 2) + 6(n-5, 3, 1^2) + 4(n-5, 2^2, 1) + 2(n-5, 2, 1^3) + (n-6, 6) + 3(n-6, 5, 1) + 4(n-6, 4, 2) + 3(n-6, 4, 1^2) + 2(n-6, 3^2) + 4(n-6, 3, 2, 1) + (n-6, 3, 1^3) + (n-6, 2^3) + (n-6, 2^2, 1^2) + (n-7, 6, 1) + (n-7, 5, 2) + (n-7, 5, 1^2) + (n-7, 4, 3) + (n-7, 4, 2, 1) + (n-7, 3^2, 1).$

43). $(n-3, 3) \times (n-4, 2^2) = (n-2, 2) + (n-2, 1^2) + 2(n-3, 3) + 3(n-3, 2, 1) + (n-3, 1^3) + (n-4, 4) + 4(n-4, 3, 1) + 3(n-4, 2^2) + 4(n-4, 2, 1^2) + (n-4, 1^4) + (n-5, 5) + 3(n-5, 4, 1) + 4(n-5, 3, 2) + 3(n-5, 3, 1^2) + 4(n-5, 2^2, 1) + 2(n-5, 2, 1^3) + (n-5, 1^5) + (n-6, 5, 1) + 2(n-6, 4, 2) + 2(n-6, 4, 1^2) + (n-6, 3^2) + 3(n-6, 3, 2, 1) + (n-6, 3, 1^3) + (n-6, 2^3) + (n-6, 2^2, 1^2) + (n-7, 5, 2) + (n-7, 4, 2, 1) + (n-7, 3, 2^2).$

44). $(n-3, 3) \times (n-4, 2, 1^2) = (n-2, 2) + (n-2, 1^2) + (n-3, 3) + 4(n-3, 2, 1) + 3(n-3, 1^3) + (n-4, 4) + 5(n-4, 3, 1) + 4(n-4, 2^2) + 7(n-4, 2, 1^2)$

$$+ 3(n-4, 1^4) + 3(n-5, 4, 1)$$
$$+ 4(n-5, 3, 2) + 7(n-5, 3, 1^2)$$
$$+ 5(n-5, 2^2, 1) + 5(n-5, 2, 1^3)$$
$$+ (n-5, 1^5) + (n-6, 5, 1)$$
$$+ 2(n-6, 4, 2) + 3(n-6, 4, 1^2)$$
$$+ (n-6, 3^2) + 4(n-6, 3, 2, 1)$$
$$+ 3(n-6, 3, 1^3) + (n-6, 2^3)$$
$$+ 2(n-6, 2^2, 1^2) + (n-6, 2, 1^4)$$
$$+ (n-7, 5, 1^2) + (n-7, 4, 2, 1)$$
$$+ (n-7, 4, 1^3) + (n-7, 3, 2, 1^2).$$

45). $(n-3, 3) \times (n-4, 1^4) = (n-3, 2, 1) + (n-3, 1^3) + (n-4, 3, 1)$
$$+ (n-4, 2^2) + 3(n-4, 2, 1^2)$$
$$+ 2(n-4, 1^4) + (n-5, 3, 2)$$
$$+ 2(n-5, 3, 1^2) + 2(n-5, 2^2, 1)$$
$$+ 3(n-5, 2, 1^3) + (n-5, 1^5)$$
$$+ (n-6, 4, 1^2) + (n-6, 3, 2, 1)$$
$$+ 2(n-6, 3, 1^3) + (n-6, 2^2, 1^2)$$
$$+ (n-6, 2, 1^4) + (n-7, 4, 1^3)$$
$$+ (n-7, 3, 1^4).$$

46). $(n-3, 2, 1) \times (n-3, 2, 1) = (n) + 2(n-1, 1) + 4(n-2, 2)$
$$+ 4(n-2, 1^2) + 5(n-3, 3)$$
$$+ 9(n-3, 2, 1) + 5(n-3, 1^3)$$
$$+ 3(n-4, 4) + 9(n-4, 3, 1)$$
$$+ 6(n-4, 2^2) + 9(n-4, 2, 1^2)$$
$$+ 3(n-4, 1^4) + (n-5, 5)$$
$$+ 4(n-5, 4, 1) + 5(n-5, 3, 2)$$
$$+ 6(n-5, 3, 1^2) + 5(n-5, 2^2, 1)$$
$$+ 4(n-5, 2, 1^3) + (n-5, 1^5)$$
$$+ (n-6, 4, 2) + (n-6, 4, 1^2)$$
$$+ (n-6, 3^2) + 2(n-6, 3, 2, 1)$$
$$+ (n-6, 3, 1^3) + (n-6, 2^3)$$
$$+ (n-6, 2^2, 1^2).$$

47). $(n-3, 2, 1) \times (n-3, 1^3) = (n-1, 1) + 2(n-2, 2) + 2(n-2, 1^2)$
$$+ 2(n-3, 3) + 5(n-3, 2, 1)$$
$$+ 2(n-3, 1^3) + (n-4, 4)$$
$$+ 4(n-4, 3, 1) + 3(n-4, 2^2)$$
$$+ 5(n-4, 2, 1^2) + 2(n-4, 1^4)$$
$$+ (n-5, 4, 1) + 2(n-5, 3, 2)$$
$$+ 3(n-5, 3, 1^2) + 3(n-5, 2^2, 1)$$
$$+ 3(n-5, 2, 1^3) + (n-5, 1^5)$$

$$+ (n-6,3,2,1) + (n-6,3,1^3)$$
$$+ (n-6,2^2,1^2) + (n-6,2,1^4).$$

48). $(n-3,2,1) \times (n-4,4) = (n-2,2) + (n-2,1^2) + 2(n-3,3)$
$+ 3(n-3,2,1) + (n-3,1^3)$
$+ 2(n-4,4) + 5(n-4,3,1)$
$+ 3(n-4,2^2) + 3(n-4,2,1^2)$
$+ 2(n-5,5) + 5(n-5,4,1)$
$+ 4(n-5,3,2) + 4(n-5,3,1^2)$
$+ 2(n-5,2^2,1) + (n-5,2,1^3)$
$+ (n-6,6) + 3(n-6,5,1)$
$+ 3(n-6,4,2) + 3(n-6,4,1^2)$
$+ (n-6,3^2) + 2(n-6,3,2,1)$
$+ (n-6,3,1^3) + (n-7,6,1)$
$+ (n-7,5,2) + (n-7,5,1^2)$
$+ (n-7,4,2,1).$

49). $(n-3,2,1) \times (n-4,3,1) = (n-1,1) + 3(n-2,2) + 3(n-2,1^2)$
$+ 5(n-3,3) + 9(n-3,2,1)$
$+ 4(n-3,1^3) + 5(n-4,4)$
$+ 13(n-4,3,1) + 8(n-4,2^2)$
$+ 11(n-4,2,1^2) + 3(n-4,1^4)$
$+ 3(n-5,5) + 10(n-5,4,1)$
$+ 11(n-5,3,2) + 12(n-5,3,1^2)$
$+ 9(n-5,2^2,1) + 6(n-5,2,1^3)$
$+ (n-5,1^5) + (n-6,6)$
$+ 4(n-6,5,1) + 6(n-6,4,2)$
$+ 6(n-6,4,1^2) + 3(n-6,3^2)$
$+ 8(n-6,3,2,1) + 4(n-6,3,1^3)$
$+ 2(n-6,2^3) + 3(n-6,2^2,1^2)$
$+ (n-6,2,1^4) + (n-7,5,2)$
$+ (n-7,5,1^2) + (n-7,4,3)$
$+ 2(n-7,4,2,1) + (n-7,4,1^3)$
$+ (n-7,3^2,1) + (n-7,3,2^2)$
$+ (n-7,3,2,1^2).$

50). $(n-3,2,1) \times (n-4,2^2) + (n-1,) + 2(n-2,2) + 2(n-2,1^2)$
$+ 3(n-3,3) + 6(n-3,2,1)$
$+ 3(n-3,1^3) + 3(n-4,4)$
$+ 8(n-4,3,1) + 5(n-4,2^2)$
$+ 8(n-4,2,1^2) + 3(n-4,1^4)$
$+ (n-5,5) + 5(n-5,4,1)$
$+ 7(n-5,3,2) + 8(n-5,3,1^2)$

$$
\begin{aligned}
&+7(n-5,2^2,1)+5(n-5,2,1^3)\\
&+(n-5,1^5)+(n-6,5,1)\\
&+3(n-6,4,2)+3(n-6,4,1^2)\\
&+2(n-6,3^2)+6(n-6,3,2,1)\\
&+3(n-6,3,1^3)+2(n-6,2^3)\\
&+3(n-6,2^2,1^2)+(n-6,2,1^4)\\
&+(n-7,4,3)+(n-7,4,2,1)\\
&+(n-7,3^2,1)+(n-7,3,2^2)\\
&+(n-7,3,2,1^2)+(n-7,2^3,1).
\end{aligned}
$$

51). $(n-3,2,1)\times(n-4,2,1^2)=(n-1,1)+3(n-2,2)+3(n-2,1^2)$

$$
\begin{aligned}
&+4(n-3,3)+9(n-3,2,1)\\
&+5(n-3,1^3)+3(n-4,4)\\
&+11(n-4,3,1)+8(n-4,2^2)\\
&+13(n-4,2,1^2)+5(n-4,1^4)\\
&+(n-5,5)+6(n-5,4,1)\\
&+9(n-5,3,2)+12(n-5,3,1^2)\\
&+11(n-5,2^2,1)+10(n-5,2,1^3)\\
&+3(n-5,1^5)+(n-6,5,1)\\
&+3(n-6,4,2)+4(n-6,4,1^2)\\
&+2(n-6,3^2)+8(n-6,3,2,1)\\
&+6(n-6,3,1^3)+3(n-6,2^3)\\
&+6(n-6,2^2,1^2)+4(n-6,2,1^4)\\
&+(n-6,1^6)+(n-7,4,2,1)\\
&+(n-7,4,1^3)+(n-7,3^2,1)\\
&+(n-7,3,2^2)+2(n-7,3,2,1^2)\\
&+(n-7,3,1^4)+(n-7,2^3,1)\\
&+(n-7,2^2,1^3).
\end{aligned}
$$

52). $(n-3,2,1)\times(n-4,1^4)=(n-2,2)+(n-2,1^2)+(n-3,3)$

$$
\begin{aligned}
&+3(n-3,2,1)+2(n-3,1^3)\\
&+3(n-4,3,1)+3(n-4,2^2)\\
&+5(n-4,2,1^2)+2(n-4,1^4)\\
&+(n-5,4,1)+2(n-5,3,2)\\
&+4(n-5,3,1^2)+4(n-5,2^2,1)\\
&+5(n-5,2,1^3)+2(n-5,1^5)\\
&+(n-6,4,1^2)+2(n-6,3,2,1)\\
&+3(n-6,3,1^3)+(n-6,2^3)\\
&+3(n-6,2^2,1^2)+3(n-6,2,1^4)\\
&+(n-6,1^6)+(n-7,3,2,1^2)\\
&+(n-7,3,1^4)+(n-7,2^2,1^3)\\
&+(n-7,2,1^5).
\end{aligned}
$$

53). $(n-3,1^3)\times(n-3,1^3)=(n)+(n-1,1)+2(n-2,2)$
$+(n-2,1^2)+2(n-3,3)$
$+2(n-3,2,1)+(n-3,1^3)$
$+(n-4,4)+2(n-4,3,1)$
$+2(n-4,2^2)+2(n-4,2,1^2)$
$+(n-4,1^4)+(n-5,3,2)$
$+(n-5,3,1^2)+2(n-5,2^2,1)$
$+2(n-5,2,1^3)+(n-5,1^5)$
$+(n-6,2^3)+(n-6,2^2,1^2)$
$+(n-6,2,1^4)+(n-6,1^6)$.

54). $(n-3,1^3)\times(n-4,4)=(n-3,2,1)+(n-3,1^3)+(n-4,4)$
$+2(n-4,3,1)+(n-4,2^2)$
$+2(n-4,2,1^2)+(n-4,1^4)$
$+2(n-5,4,1)+(n-5,3,2)$
$+3(n-5,3,1^2)+(n-5,2^2,1)$
$+(n-5,2,1^3)+(n-6,5,1)$
$+(n-6,4,2)+2(n-6,4,1^2)$
$+(n-6,3,2,1)+(n-6,3,1^3)$
$+(n-7,5,1^2)+(n-7,4,1^3)$.

55). $(n-3,1^3)\times(n-4,3,1)=(n-2,2)+(n-2,1^2)+2(n-3,3)$
$+4(n-3,2,1)+2(n-3,1^3)$
$+2(n-4,4)+6(n-4,3,1)$
$+4(n-4,2^2)+6(n-4,2,1^2)$
$+2(n-4,1^4)+(n-5,5)$
$+4(n-5,4,1)+5(n-5,3,2)$
$+6(n-5,3,1^2)+5(n-5,2^2,1)$
$+4(n-5,2,1^3)+(n-5,1^5)$
$+(n-6,5,1)+2(n-6,4,2)$
$+3(n-6,4,1^2)+(n-6,3^2)$
$+4(n-6,3,2,1)+3(n-6,3,1^3)$
$+(n-6,2^3)+2(n-6,2^2,1^2)$
$+(n-6,2,1^4)+(n-7,4,2,1)$
$+(n-7,4,1^3)+(n-7,3,2,1^2)$
$+(n-7,3,1^4)$.

56). $(n-3,1^3)\times(n-4,2^2)=(n-2,2)+(n-2,1^2)+(n-3,3)$
$+3(n-3,2,1)+2(n-3,1^3)$
$+(n-4,4)+4(n-4,3,1)$
$+3(n-4,2^2)+4(n-4,2,1^2)$
$+(n-4,1^4)+2(n-5,4,1)$
$+3(n-5,3,2)+5(n-5,3,1^2)$

$$+3(n-5,2^2,1)+3(n-5,2,1^3)$$
$$+(n-6,4,2)+(n-6,4,1^2)$$
$$+(n-6,3^2)+3(n-6,3,2,1)$$
$$+2(n-6,3,1^3)+(n-6,2^3)$$
$$+2(n-6,2^2,1^2)+(n-6,2,1^4)$$
$$+(n-7,3^2,1)+(n-7,3,2,1^2)$$
$$+(n-7,2^2,1^3).$$

57). $(n-3,1^3)\times(n-4,2,1^2)=(n-1,1)+2(n-2,2)+2(n-2,1^2)$
$$+3(n-3,3)+5(n-3,2,1)$$
$$+2(n-3,1^3)+2(n-4,4)$$
$$+6(n-4,3,1)+4(n-4,2^2)$$
$$+6(n-4,2,1^2)+2(n-4,1^4)$$
$$+(n-5,5)+3(n-5,4,1)$$
$$+5(n-5,3,2)+5(n-5,3,1^2)$$
$$+6(n-5,2^2,1)+5(n-5,2,1^3)$$
$$+2(n-5,1^5)+(n-6,4,2)$$
$$+(n-6,4,1^2)+(n-6,3^2)$$
$$+4(n-6,3,2,1)+3(n-6,3,1^3)$$
$$+2(n-6,2^3)+4(n-6,2^2,1^2)$$
$$+3(n-6,2,1^4)+(n-6,1^6)$$
$$+(n-7,3,2^2)+(n-7,3,2,1^2)$$
$$+(n-7,3,1^4)+(n-7,2^3,1)$$
$$+(n-7,2^2,1^3)+(n-7,2,1^5).$$

58). $(n-3,1^3)\times(n-4,1^4)=(n-1,1)+(n-2,2)+(n-2,1^2)$
$$+(n-3,3)+2(n-3,2,1)$$
$$+(n-3,1^3)+(n-4,4)$$
$$+2(n-4,3,1)+(n-4,2^2)$$
$$+2(n-4,2,1^2)+(n-4,1^4)$$
$$+(n-5,4,1)+(n-5,3,2)$$
$$+2(n-5,3,1^2)+2(n-5,2^2,1)$$
$$+2(n-5,2,1^3)+(n-5,1^5)$$
$$+(n-6,3,2,1)+(n-6,3,1^3)$$
$$+(n-6,2^3)+2(n-6,2^2,1^2)$$
$$+2(n-6,2,1^4)+(n-6,1^6)$$
$$+(n-7,2^3,1)+(n-7,2^2,1^3)$$
$$+(n-7,2,1^5)+(n-7,1^7).$$

---

3. Tables furnishing the analysis of the Kronecker products for $n \leqq 8$. Since the product of any irreducible representation (λ) of the symmetric group on n letters by the alternating representation (1^n) is the associated irreducible

representation $(\lambda)^*$ the product $(\lambda) \times (\mu)$ is the associate of the product $(\lambda) \times (\mu)^*$ or, equivalently, the associate of the product $(\lambda)^* \times (\mu)$. In fact these statements merely reflect the fact that Kronecker multiplication is associative:

$$(\lambda) \times \{(\mu) \times (\nu)\} = \{(\lambda) \times (\mu)\} \times (\nu).$$

For when $(\nu) = (1^n)$ the left-hand side of this equality is $(\lambda) \times (\mu)^*$ whilst the right-hand side is $\{(\lambda) \times (\mu)\}^*$. As a corollary we have $(\lambda)^* \times (\mu)^* = (\lambda) \times (\mu)$; for the left-hand side $= \{(\lambda)^* \times (\mu)\}^* = \{(\lambda) \times (\mu)\}^{**} = (\lambda) \times (\mu)$, the operation of taking the associate of a representation being reflexive. These facts shorten considerably the following tables. The coefficients appear in the body of the tables; if we enter the tables on the left with a product $(\lambda) \times (\mu)$ the representations which are multiplied by the coefficients are found at the *head* of the table. The product $(\lambda) \times (\mu)^*$ is found at the same place in the table but at the right rather than the left and the representations which are multiplied by the coefficients occurring in the table are found at the foot of the table. If a product $(\lambda) \times (\mu)$ is not found in the table either $(\lambda)^* \times (\mu)^*$, which is the same as $(\lambda) \times (\mu)$, or $(\lambda)^* \times (\mu)$, which is the associate of $(\lambda) \times (\mu)$, will be found. The general formulae already given furnish all products up to $n \leqq 6$ and all products for $n = 7$ save $(3^2, 1) \times (3^2, 1)$ and its associate $(3^2, 1) \times (3, 2^2)$. From the symmetry properties of the coefficient symbol $g^{(\nu)}_{(\lambda)(\mu)}$ all coefficients in the analysis of $(3^2, 1) \times (3^2, 1)$ are known, from the other products in the table for $n = 7$, save $c_{3^2,1}$ and $c_{3,2^2}$. These are determined from the character table for $n = 7$ the sum being determined by an even class and the difference by an odd class. For $n = 8$ the formulae given determine all products save those from $(4^2) \times (4^2)$ to the end of the table. The following detail of the analysis of this product will show clearly how all analyses of products which are not furnished by the formulae of section **2** may be obtained. From the symmetry properties of the coefficient symbol we have

$$\begin{aligned}(4^2) \times (4^2) = (8) &+ (6, 2) + (5, 1^3) + c_{4^2}(4^2) + c_{4,3,1}(4, 3, 1) + c_{4,2^2}(4, 2^2) \\ &+ c_{4,2,1^2}(4, 2, 1^2) + c_{3^2,2}(3^2, 2) + c_{3^2,1^2}(3^2, 1^2) \\ &+ c_{3,2^2,1}(3, 2^2, 1) + c_{2^4}(2^4).\end{aligned}$$

From the table of characters (**5**, p. 179) for the symmetric group on 8 letters we find

(a) from the class $(1, 7)$, $c_{4,2,1^2} = 0$.

(b) from the class $(2, 6)$, $\alpha - \beta = 2$ where $\alpha = c_{4,2^2} + c_{3^2,1^2}$, $\beta = c_{4,3,1} + c_{3,2^2,1}$.

(c) from the class $(3, 5)$ $\alpha - \gamma = c_{3^2,2}$ where $\gamma = c_{4^2} + c_{2^4}$.

(d) from the class (2^4), $3\gamma - \beta + 4\alpha - 6c_{3^2,2} = 14$.

(e) from the class (1^8), $\gamma + 5\beta + 4\alpha + c_{3^2,2} = 10$.

Hence $\alpha = 2$, $\beta = 0$, $\gamma = 2$, $c_{3^2,2} = 0$.

(f) from the class $(1^6, 2)$ we find $7\gamma' + 5\beta' + 2\alpha' = 0$ where
$\alpha' = c_{4,2^2} - c_{3^2,1^2}$; $\beta' = c_{4,3,1} - c_{3,2^2,1}$; $\gamma' = c_{4^2} - c_{2^4}$.

(g) from the class $(1^4, 4)$ we find $\gamma' + 2\beta' = 0$.

(h) from the class $(2, 3^2)$ we find $\alpha' + \beta' - 2\gamma' = 0$.
Hence $\alpha' = 0 = \beta' = \gamma'$ so that $c_{4,2^2} = c_{3^2,2} = 1$; $c_{4,3,1} = c_{3,2^2,1} = 0$; $c_{4^2} = c_{2^4} = 1$ and we have the analysis given in the table.

In the following tables the products by the identity representation (n), which acts as a unit, and by the alternating representation (1^n), which sends each representation into its associate, are omitted. Thus the table for $n = 2$ is not given.

1. $n = 3$.

$$(2,1) \times (2,1) = (3) + (2,1) + (1^3).$$

2. $n = 4$.

	(4)	$(3,1)$	(2^2)	$(2,1^2)$	(1^4)	
$(3,1)\times(3,1)$	1	1	1	1		$(3,1)\times(2,1^2)$
$(3,1)\times(2^2)$		1		1		$(3,1)\times(2^2)$
$(2^2)\times(2^2)$	1		1		1	$(2^2)\times(2^2)$
	(1^4)	$(2,1^2)$	(2^2)	$(3,1)$	(4)	

3. $n = 5$.

	(5)	$(4,1)$	$(3,2)$	$(3,1^2)$	$(2^2,1)$	$(2,1^3)$	(1^5)	
$(4,1)\times(4,1)$	1	1	1	1				$(4,1)\times(2,1^3)$
$(4,1)\times(3,2)$		1	1	1	1			$(4,1)\times(2^2,1)$
$(4,1)\times(3,1^2)$		1	1	1	1	1		$(4,1)\times(3,1^2)$
$(3,2)\times(3,2)$	1	1	1	1	1	1		$(3,2)\times(2^2,1)$
$(3,2)\times(3,1^2)$		1	1	2	1	1		$(3,2)\times(3,1^2)$
$(3,1^2)\times(3,1^2)$	1	1	2	1	2	1	1	$(3,1^2)\times(3,1^2)$
	(1^5)	$(2,1^3)$	$(2^2,1)$	$(3,1^2)$	$(3,2)$	$(4,1)$	(5)	

4. $n = 6$.

	(6)	$(5,1)$	$(4,2)$	$(4,1^2)$	(3^2)	$(3,2,1)$	$(3,1^3)$	(2^3)	$(2^2,1^2)$	$(2,1^4)$	(1^6)	
$(5,1)\times(5,1)$	1	1	1	1								$(5,1)\times(2,1^4)$
$(5,1)\times(4,2)$		1	1	1	1	1						$(5,1)\times(2^2,1^2)$
$(5,1)\times(4,1^2)$		1	1	1		1	1					$(5,1)\times(3,1^3)$
$(5,1)\times(3^2)$			1			1						$(5,1)\times(2^3)$
$(5,1)\times(3,2,1)$			1	1	1	2	1	1	1			$(5,1)\times(3,2,1)$
$(4,2)\times(4,2)$	1	1	2	1		2	1	1				$(4,2)\times(2^2,1^2)$
$(4,2)\times(4,1^2)$		1	1	2	1	2	1		1			$(4,2)\times(3,1^3)$
$(4,2)\times(3^2)$		1		1	1	1			1			$(4,2)\times(2^3)$
$(4,2)\times(3,2,1)$		1	2	2	1	3	2	1	2	1		$(4,2)\times(3,2,1)$
$(4,1^2)\times(4,1^2)$	1	1	2	1	1	2	1	1	1	1		$(4,1^2)\times(3,1^3)$
$(4,1^2)\times(3^2)$			1	1		1	1	1				$(4,1^2)\times(2^3)$
$(4,1^2)\times(3,2,1)$		1	2	2	1	4	2	1	2	1		$(4,1^2)\times(3,2,1)$
$(3^2)\times(3^2)$	1		1				1	1				$(3^2)\times(2^3)$
$(3^2)\times(3,2,1)$		1	1	1		2	1		1	1		$(3^2)\times(3,2,1)$
$(3,2,1)\times(3,2,1)$	1	2	3	4	2	5	4	2	3	2	1	$(3,2,1)\times(3,2,1)$
	(1^6)	$(2,1^4)$	$(2^2,1^2)$	$(3,1^3)$	(2^3)	$(3,2,1)$	$(4,1^3)$	(3^3)	$(4,2)$	$(5,1)$	(6)	

5. $n = 7$.

	(7)	$(6,1)$	$(5,2)$	$(5,1^2)$	$(4,3)$	$(4,2,1)$	$(4,1^3)$	$(3^2,1)$	$(3,2^2)$	$(3,2,1^2)$	$(3,1^4)$	$(2^3,1)$	$(2^2,1^3)$	$(2,1^5)$	(1^7)	
$(6,1)\times(6,1)$	1	1	1	1												$(6,1)\times(2,1^5)$
$(6,1)\times(5,2)$		1	1	1	1	1										$(6,1)\times(2^2,1^3)$
$(6,1)\times(5,1^2)$		1	1	1		1	1									$(6,1)\times(3,1^4)$
$(6,1)\times(4,3)$			1		1	1		1								$(6,1)\times(2^3,1)$
$(6,1)\times(4,2,1)$			1	1	1	2	1	1	1	1						$(6,1)\times(3,2,1^2)$
$(6,1)\times(4,1^3)$				1		1	1			1	1					$(6,1)\times(4,1^3)$
$(6,1)\times(3^2,1)$					1	1		1	1	1						$(6,1)\times(3,2^2)$
$(5,2)\times(5,2)$	1	1	2	1	1	2	1	1	1							$(5,2)\times(2^2,1^3)$
$(5,2)\times(5,1^2)$		1	1	2	1	2	1	1		1						$(5,2)\times(3,1^4)$
$(5,2)\times(4,3)$		1	1	1	1	2		1	1	1						$(5,2)\times(2^3,1)$
$(5,2)\times(4,2,1)$		1	2	2	2	4	2	2	2	3	1	1				$(5,2)\times(3,2,1^2)$
$(5,2)\times(4,1^3)$			1	1		2	2	1	1	2	1		1			$(5,2)\times(4,1^3)$
$(5,2)\times(3^2,1)$			1	1	1	2	1	2	1	2		1	1			$(5,2)\times(3,2^2)$
$(5,1^2)\times(5,1^2)$	1	1	2	1	1	2	1		1	1	1					$(5,1^2)\times(3,1^4)$
$(5,1^2)\times(4,3)$			1	1	1	2	1	1	1	1						$(5,1^2)\times(2^3,1)$
$(5,1^2)\times(4,2,1)$		1	2	2	2	4	2	3	2	3	1	1	1			$(5,1^2)\times(3,2,1^2)$
$(5,1^2)\times(4,1^3)$		1	1	1	1	2	1	1	1	2	1	1	1	1		$(5,1^2)\times(4,1^3)$
$(5,1^2)\times(3^2,1)$			1		1	3	1	1	2	2	1	1				$(5,1^2)\times(3,2^2)$
$(4,3)\times(4,3)$	1	1	1	1	1	1	1	1	1	1		1				$(4,3)\times(2^3,1)$
$(4,3)\times(4,2,1)$		1	2	2	1	4	2	2	2	3	1	1	1			$(4,3)\times(3,2,1^2)$
$(4,3)\times(4,1^3)$				1	1	2	2	1	1	2	1	1				$(4,3)\times(4,1^3)$
$(4,3)\times(3^2,1)$		1	1	1	1	2	1	1	1	2	1	1	1			$(4,3)\times(3,2^2)$
$(4,2,1)\times(4,2,1)$	1	2	4	4	4	9	5	5	5	8	3	3	3	1		$(4,2,1)\times(3,2,1^2)$
$(4,2,1)\times(4,1^3)$		1	2	2	2	5	2	3	3	5	2	2	2	1		$(4,2,1)\times(4,1^3)$
$(4,2,1)\times(3^2,1)$		1	2	3	2	5	3	3	3	5	2	2	2	1		$(4,2,1)\times(3,2^2)$
$(4,1^3)\times(4,1^3)$	1	1	2	1	2	2	1	2	2	2	1	2	2	1		$(4,1^3)\times(4,1^3)$
$(4,1^3)\times(3^2,1)$			1	1	1	3	2	2	2	3	1	1	1			$(4,1^3)\times(3,2^2)$
$(3^2,1)\times(3^2,1)$	1	1	2	1	1	3	2	1	2	3	2	1	1	1		$(3^2,1)\times(3,2^2)$
	(1^7)	$(2,1^5)$	$(2^2,1^3)$	$(3,1^4)$	$(2^3,1)$	$(3,2,1^2)$	$(4,1^3)$	$(3,2^2)$	$(3^2,1)$	$(4,2,1)$	$(5,1^2)$	$(4,3)$	$(5,2)$	$(6,1)$	(7)	

6. $n = 8$.

	(8)	$(7,1)$	$(6,2)$	$(6,1^2)$	$(5,3)$	$(5,2,1)$	$(5,1^3)$	(4^2)	$(4,3,1)$	$(4,2^2)$	$(4,2,1^2)$	$(4,1^4)$	$(3^2,2)$	$(3^2,1^2)$	$(3,2^2,1)$	$(3,2,1^3)$	$(3,1^5)$	(2^4)	$(2^3,1^2)$	$(2^2,1^4)$	$(2,1^6)$	(1^8)	
$(7,1)\times(7,1)$	1	1	1	1																			$(7,1)\times(2,1^6)$
$(7,1)\times(6,2)$		1	1	1	1	1																	$(7,1)\times(2^2,1^4)$
$(7,1)\times(6,1^2)$		1	1	1		1	1																$(7,1)\times(3,1^5)$
$(7,1)\times(5,3)$			1		1	1		1	1														$(7,1)\times(2^3,1^2)$
$(7,1)\times(5,2,1)$			1	1	1	2	1		1	1	1												$(7,1)\times(3,2,1^3)$
$(7,1)\times(5,1^3)$				1		1	1				1	1											$(7,1)\times(4,1^4)$
$(7,1)\times(4^2)$					1				1														$(7,1)\times(2^4)$
$(7,1)\times(4,3,1)$					1	1		1	2	1	1		1	1									$(7,1)\times(3,2^2,1)$
$(7,1)\times(4,2^2)$						1			1	1	1		1		1								$(7,1)\times(3^2,1^2)$
$(7,1)\times(4,2,1^2)$						1	1		1	1	2	1		1	1	1							$(7,1)\times(4,2,1^2)$
$(7,1)\times(3^2,2)$									1	1			1	1	1								$(7,1)\times(3^2,2)$
$(6,2)\times(6,2)$	1	1	2	1	1	2	1	1	1	1													$(6,2)\times(2^2,1^4)$
$(6,2)\times(6,1^2)$		1	1	2	1	2	1		1		1												$(6,2)\times(3,1^5)$
$(6,2)\times(5,3)$		1	1	1	2	2			2	1	1		1										$(6,2)\times(2^3,1^2)$
$(6,2)\times(5,2,1)$		1	2	2	2	4	2	1	3	2	3	1	1	1	1								$(6,2)\times(3,2,1^3)$
$(6,2)\times(5,1^3)$			1	1		2	2		1	1	2	1		1		1							$(6,2)\times(4,1^4)$
$(6,2)\times(4^2)$			1			1		1	1	1				1									$(6,2)\times(2^4)$
$(6,2)\times(4,3,1)$			1	1	2	3	1	1	4	2	3		2	2	2	1							$(6,2)\times(3,2^2,1)$
$(6,2)\times(4,2^2)$			1		1	2	1	1	2	3	2	1	1	2	2	1		1					$(6,2)\times(3^2,1^2)$
$(6,2)\times(4,2,1^2)$				1	1	3	2		3	2	5	2	2	2	3	3	1		1				$(6,2)\times(4,2,1^2)$
$(6,2)\times(3^2,2)$					1	1			2	1	2		2	1	2	1			1				$(6,2)\times(3^2,2)$
$(6,1^2)\times(6,1^2)$	1	1	2	1	1	2	1			1	1	1											$(6,1^2)\times(3,1^5)$
	(1^8)	$(2,1^6)$	$(2^2,1^4)$	$(3,1^5)$	$(2^3,1^2)$	$(3,2,1^3)$	$(4,1^4)$	(2^4)	$(3,2^2,1)$	$(3^2,1^2)$	$(4,2,1^2)$	$(5,1^3)$	$(3^2,2)$	$(4,2^2)$	$(4,3,1)$	$(5,2,1)$	$(6,1^2)$	(4^2)	$(5,3)$	$(6,2)$	$(7,1)$	(8)	

→

	(8)	(7, 1)	(6, 2)	$(6, 1^2)$	(5, 3)	(5, 2, 1)	$(5, 1^3)$	(4^2)	(4, 3, 1)	$(4, 2^2)$	$(4, 2, 1^2)$	$(4, 1^4)$	$(3^2, 2)$	$(3^2, 1^2)$	$(3, 2^2, 1)$	$(3, 2, 1^3)$	$(3, 1^5)$	(2^4)	$(2^3, 1^2)$	$(2^2, 1^4)$	$(2, 1^6)$	(1^8)	
$(6, 1^2) \times (5, 3)$			1	1	1	2	1	1	2	1	1			1									$(6, 1^2) \times (2^3, 1^2)$
$(6, 1^2) \times (5, 2, 1)$		1	2	2	2	4	2	1	3	2	3	1	1	1	1	1							$(6, 1^2) \times (3, 2, 1^3)$
$(6, 1^2) \times (5, 1^3)$		1	1	1	1	2	1		1	1	2	1			1	1	1						$(6, 1^2) \times (4, 1^4)$
$(6, 1^2) \times (4^2)$					1	1			1		1		1										$(6, 1^2) \times (2^4)$
$(6, 1^2) \times (4, 3, 1)$			1		2	3	1	1	4	3	3	1	2	2	2	1							$(6, 1^2) \times (3, 2^2, 1)$
$(6, 1^2) \times (4, 2^2)$				1	1	2	1		3	1	3		2	2	2	1			1				$(6, 1^2) \times (3^2, 1^2)$
$(6, 1^2) \times (4, 2, 1^2)$			1	1	1	3	2	1	3	3	4	2	2	3	3	3	1	1	1	1			$(6, 1^2) \times (4, 2, 1^2)$
$(6, 1^2) \times (3^2, 2)$						1		1	2	2	2		1	2	2	1		1					$(6, 1^2) \times (3^2, 2)$
$(5, 3) \times (5, 3)$	1	1	2	1	1	2	1	1	2	2	1		1	1	1								$(5, 3) \times (2^3, 1^2)$
$(5, 3) \times (5, 2, 1)$		1	2	2	2	5	2	1	4	3	4	1	2	2	2	1							$(5, 3) \times (3, 2, 1^3)$
$(5, 3) \times (5, 1^3)$				1	1	2	2		2	1	3	1	1	1	1	1							$(5, 3) \times (4, 1^4)$
$(5, 3) \times (4^2)$		1		1	1	1			1		1		1		1								$(5, 3) \times (2^4)$
$(5, 3) \times (4, 3, 1)$		1	2	2	2	4	2	1	4	3	4	1	2	3	3	2		1	1				$(5, 3) \times (3, 2^2, 1)$
$(5, 3) \times (4, 2^2)$			1	1	2	3	1		3	2	4	1	2	1	3	2	1		1				$(5, 3) \times (3^2, 1^2)$
$(5, 3) \times (4, 2, 1^2)$			1	1	1	4	3	1	4	4	6	3	2	4	4	4	1	1	1	1			$(5, 3) \times (4, 2^2, 1)$
$(5, 3) \times (3^2, 2)$			1		1	2	1	1	2	2	2	1	1	2	2	2		1	1	1			$(5, 3) \times (3^2, 2)$
$(5, 2, 1) \times (5, 2, 1)$	1	2	4	4	5	9	5	2	9	6	9	3	4	5	5	4	1	1	1				$(5, 2, 1) \times (3, 2, 1^3)$
$(5, 2, 1) \times (5, 1^3)$		1	2	2	2	5	2	1	4	3	5	2	2	3	3	3	1		1	1			$(5, 2, 1) \times (4, 1^4)$
$(5, 2, 1) \times (4^2)$			1	1	1	2	1		2	2	2		1	1	1	1							$(5, 2, 1) \times (2^4)$
$(5, 2, 1) \times (4, 3, 1)$		1	3	3	4	9	4	2	9	7	10	3	5	6	7	5	1	1	2	1			$(5, 2, 1) \times (3, 2^2, 1)$
$(5, 2, 1) \times (4, 2^2)$		1	2	2	3	6	3	2	7	5	8	3	4	5	6	5	1	1	2	1			$(5, 2, 1) \times (3^2, 1^2)$
$(5, 2, 1) \times (4, 2, 1^2)$		1	3	3	4	9	5	2	10	8	13	5	6	8	10	9	3	2	4	3	1		$(5, 2, 1) \times (4, 2, 1^2)$
	(1^8)	$(2, 1^6)$	$(2^2, 1^4)$	$(3, 1^5)$	$(2^3, 1^2)$	$(3, 2, 1^3)$	$(4, 1^4)$	(2^4)	$(3, 2^2, 1)$	$(3^2, 1^2)$	$(4, 2, 1^2)$	$(5, 1^3)$	$(3^2, 2)$	$(4, 2^2)$	(4, 3, 1)	(5, 2, 1)	$(6, 1^2)$	(4^2)	(5, 3)	(6, 2)	(7, 1)	(8)	

←

	(8)	(7, 1)	(6, 2)	$(6, 1^2)$	(5, 3)	(5, 2, 1)	$(5, 1^3)$	(4^2)	(4, 3, 1)	$(4, 2^2)$	$(4, 2, 1^2)$	$(4, 1^4)$	$(3^2, 2)$	$(3^2, 1^2)$	$(3, 2^2, 1)$	$(3, 2, 1^3)$	$(3, 1^5)$	(2^4)	$(2^3, 1^2)$	$(2^2, 1^4)$	$(2, 1^6)$	(1^8)	
$(5, 2, 1) \times (3^2, 2)$			1	1	2	4	2	1	5	4	6	2	3	4	5	4	1	1	2	1			$(5, 2, 1) \times (3^2, 2)$
$(5, 1^3) \times (5, 1^3)$	1	1	2	1	2	2	1	1	2	2	2	1	1	1	2	2	1	1	1	1	1		$(5, 1^3) \times (4, 1^4)$
$(5, 1^3) \times (4^2)$						1	1	1	1	1	1	1		1	1								$(5, 1^3) \times (2^4)$
$(5, 1^3) \times (4, 3, 1)$			1	1	2	4	2	1	5	4	6	2	3	3	4	3	1	1	1				$(5, 1^3) \times (3, 2^2, 1)$
$(5, 1^3) \times (4, 2^2)$			1	1	1	3	2	1	4	3	4	1	2	4	3	3		1	1	1			$(5, 1^3) \times (3^2, 1^2)$
$(5, 1^3) \times (4, 2, 1^2)$		1	2	2	3	5	2	1	6	4	6	2	4	4	6	5	2	1	3	2	1		$(5, 1^3) \times (4, 2, 1^2)$
$(5, 1^3) \times (3^2, 2)$					1	2	1		3	2	4	1	2	2	3	2			1				$(5, 1^3) \times (3^2, 2)$
$(4^2) \times (4^2)$	1		1				1	1		1				1				1					$(4^2) \times (2^4)$
$(4^2) \times (4, 3, 1)$		1	1	1	1	2	1		2	1	2	1	1	1	2	1			1				$(4^2) \times (3, 2^2, 1)$
$(4^2) \times (4, 2^2)$			1			2	1	1	1	2	1	1		2	1	1		1		1			$(4^2) \times (3^2, 1^2)$
$(4^2) \times (4, 2, 1^2)$				1	1	2	1		2	1	4	1	2	1	2	2	1		1				$(4^2) \times (4, 2, 1^2)$
$(4^2) \times (3^2, 2)$				1	1	1			1		2		1		1	1	1		1				$(4^2) \times (3^2, 2)$
$(4, 3, 1) \times (4, 3, 1)$	1	2	4	4	4	9	5	2	8	7	11	4	5	6	8	7	2	2	3	2			$(4, 3, 1) \times (3, 2^2, 1)$
$(4, 3, 1) \times (4, 2^2)$		1	2	3	3	7	4	1	7	5	9	3	4	5	6	6	2	1	3	2	1		$(4, 3, 1) \times (3^2, 1^2)$
$(4, 3, 1) \times (4, 2, 1^2)$		1	3	3	4	10	6	2	11	9	14	6	6	9	11	10	3	2	4	3	1		$(4, 3, 1) \times (4, 2, 1^2)$
$(4, 3, 1) \times (3^2, 2)$		1	2	2	2	5	3	1	5	4	6	3	2	4	5	5	2	1	2	2	1		$(4, 3, 1) \times (3^2, 2)$
$(4, 2^2) \times (4, 2^2)$	1	1	3	1	2	5	3	2	5	6	6	4	2	5	5	5	2	2	1	2			$(4, 2^2) \times (3^2, 1^2)$
$(4, 2^2) \times (4, 2, 1^2)$		1	2	3	4	8	4	1	9	6	12	4	6	6	9	8	3	1	4	2	1		$(4, 2^2) \times (4, 2, 1^2)$
$(4, 2^2) \times (3^2, 2)$		1	1	2	2	4	2		4	2	6	2	3	2	4	4	2		2	1	1		$(4, 2^2) \times (3^2, 2)$
$(4, 2, 1^2) \times (4, 2, 1^2)$	1	2	5	4	6	13	6	4	14	12	17	6	8	12	14	13	4	4	6	5	2	1	$(4, 2, 1^2) \times (4, 2, 1^2)$
$(4, 2, 1^2) \times (3^2, 2)$			2	2	2	6	4	2	6	6	8	4	4	6	6	6	2	2	2	2			$(4, 2, 1^2) \times (3, 2^2)$
$(3^2, 2) \times (3^2, 2)$	1	1	2	1	1	3	2	1	2	3	4	2	1	3	2	3	1	1	1	2	1	1	$(3^2, 2) \times (3^2, 2)$
	(1^8)	$(2, 1^6)$	$(2^2, 1^4)$	$(3, 1^5)$	$(2^3, 1^2)$	$(3, 2, 1^3)$	$(4, 1^4)$	(2^4)	$(3, 2^2, 1)$	$(3^2, 1^2)$	$(4, 2, 1^2)$	$(5, 1^3)$	$(3^2, 2)$	$(4, 2^2)$	(4, 3, 1)	(5, 2, 1)	$(6, 1^2)$	(4^2)	(5, 3)	(6, 2)	(7, 1)	(8)	

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